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Refinement of the partial adjustment model using continuous-time econometrics

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Abstract in English

This paper presents some suggestions for the specification of dynamic models. These suggestions are based on the supposed continuous-time nature of most economic processes. In particular, the partial adjustment model –or Koyck lag model– is discussed.
The refinement of this model is derived from the continuous-time econometric literature. We find three alternative formulas for this refinement, depending on the particular econometric literature which is used. Two of these formulas agree with an intuitive example.
In passing, it is shown that that the continuous-time models of Sims and Bergstrom are closely related. Also the inverse of Bergstrom's approximate analog has been introduced, making use of engineering mathematics.

Abstract in Dutch

Dit Discussion Paper presenteert enige suggesties voor de specificatie van dynamische modellen. Deze suggesties zijn gebaseerd op de veronderstelling dat economische processen zich in continue tijd afspelen. In het bijzonder wordt besproken het model van de geleidelijke aanpassing, of Koyck model.
De verfijning van dit model is afgeleid van de econometrische literatuur over continue tijd. We vinden drie alternatieve formules voor deze verfijning, afhankelijk van welke econometrische literatuur wordt gebruikt. Twee van deze formules komen overeen met een intuitief voorbeeld.
In het voorbijgaan wordt aangetoond dat de modellen in continue tijd van Sims en Bergstrom nauw verwant zijn. Ook wordt de inverse van Bergstrom's benaderend analogon geïntroduceerd, gebruik makend van ingenieurswiskunde.

Contents

Summary

Econometric modeling often involves some dynamics, in the form of lagged variables, or differences over time, etcetera. Unfortunately, economic theory may tell us what variables are relevant, but is usually only marginally informative about their dynamics.

This paper explores a guideline for the specifications of dynamics, derived from a very general principle: the nature of time. Very few economic processes run intermittently, with the annual or quarterly frequency of empirical data series. Most proceed continuously from day to day. Production, consumption, and investments are not done at January 1, or the first day in each quarter, but are spread over time.

This simple fact has been the subject of much research; both about the problems involved in ignoring this fact, and about the estimation of continuous-time models. Here we ask: can we use this research to find simple and practical recipes for the specification of dynamic economic relations, in discrete time?

In particular, the partial adjustment model -or Koyck lag model- is studied. Here the dependent variable adapts gradually to changes in the independent variable, using the lagged value of the dependent variable. An alternative form of this model is the exponential distributed lag model. The limiting case of this model is infinitely slow adaptation, with amounts to the dependent variable being the cumulated independent variable.

First, the problem is discussed using a simple intuitive example of the limiting "cumulative" case: capital accumulation. Assuming investment takes place continually, the form of the relation between annual production capacity and annual investment is discussed.

Next, the continuous-time econometric literature is used to derive results more formally. The work of Sims (1971) is applied to our partial adjustment model in the distributed lag form. The models of Bergstrom are applied to our partial adjustment model in the usual difference equation form.

The result is a simple suggestion: the partial adjustment model should be refined by adding a term with the lagged independent variable to the right-hand side. The specific form of this term varies with the approach chosen. In the simplest case, the lagged and the current independent variable have the same coefficient. Two more complicated formulas for the coefficient of the lagged independent are suggested. One of them, derived from the widely used quadratic exact discrete-time analog of Bergstrom conflicts with the intuitive investment example.

In doing so, it has been shown that continuous-time models of Sims and Bergstrom are closely related. Also the inverse of Bergstrom's approximate analog has been introduced, making use of engineering mathematics. This analog is known in the engineering literature, where it is expressed in entirely different notation.

1 Introduction

Econometric modeling often involves some dynamics, in the form of lagged variables, or differences over time, etc. Unfortunately economic theory may tell us what variables are relevant, but is usually only marginally informative about their dynamics. This paper explores a guideline for the specifications of the dynamics, derived from a very general principle: the nature of time.

Very few economic processes run intermittently, with the annual or quarterly frequency of empirical data series. Most proceed continuously from day to day. Production, consumption, and investments are not done at January 1, or the first day in each quarter, but are spread over time.

This simple fact has been the subject of much research; both about the problems involved in ignoring this fact, and about the estimation of continuous-time models. See for instance the references in Ten Cate (1993) and in the more recent McCrorie and Chambers (2004). Here we ask: can we use this research to find simple and practical recipes for the specification of dynamic economic relations, in discrete time? This question is answered affirmative for the partial adjustment model.

In the next section an intuitive example is given, showing the effect of continuous time modeling on a dynamic equation in discrete time. Section 3 discusses details about continuous time.

In sections 4 through 7 refinements of the partial adjustment model

$$Y_t = \rho Y_{t-1} + \phi X_t \tag{1.1}$$

are derived from an assumed underlying continuous-time model, using two approaches:

- In section 4 the continuous-time analysis of distributed lags of Sims (1971) is applied to the partial adjustment model in the form of a distributed lag model.
- In sections 5 through 7 the analysis of differential equations as initiated in Bergstrom (1966) is applied to the partial adjustment model in the form of a difference equation.

It is shown that these two approaches are closely related.

Section 8 discusses the reverse problem: what underlying continuous-time model is implied by the partial adjustment model (1.1) without these refinements?

Section 9 sums up the results. Mathematical derivations are put into three Appendices. In Appendix B a relation between engineering mathematics and continuous-time econometrics is used, which seems to have been overlooked in the literature.

2 An intuitive example

As an example we consider the production capacity Y_t created by investment in equipment X_t. For simplicity, the investments are assumed to be immediately usable for production, without gestation lag.

Then the model equation for the production capacity should take into account only one half of the investment figure of the current time period. Why? With all current period investment in the equation, the production capacity would represent the stock of capital accumulated at the *end* of the period. On the other hand, a zero coefficient for current period investment would represent the stock of capital at the *beginning* of the period. Otherwise stated: since on the average the current period has elapsed for one half, the investment during the current period must be taken into account for one half.

This can be written as:

$$Y_t = \frac{1}{\kappa}\left(\frac{1}{2}X_t + X_{t-1} + X_{t-2} + \cdots\right) \tag{2.1}$$

with capital coefficient κ and (net) investments X_t. Thus the lag distribution of the investments in the equation for production capacity has a peculiar shape at the near end, due to the assumption that (macro) investments flow continually throughout the time period. Several macro-economic models of the CPB have an equation for production capacity in this form. This example will recur repeatedly in the formal analysis below.

Individual investment projects are usually not evenly distributed within the discrete time periods of the data series. If it were known that such projects are usually late/early within the time periods, then the above coefficient of one half should be somewhat smaller/larger. In the absence of such knowledge, however, the above equation is the most likely.

In the sequel we shall discuss in detail what time pattern of the investments is required for (2.1) to hold exactly; as we shall see at the end of section 4 below, it is not necessary that the investments are constant within the discrete time periods.

The approaches discussed below are based on various assumptions about the time pattern of the exogenous variable X_t. These assumptions may or may not be very realistic. *However, ignoring the discussion in this paper altogether also implies some assumption about this pattern, which might be entirely unrealistic.* For example, to base the production capacity in (2.1) on the summation of *all* investments implies that investments are indeed always done at January 1, or –in a quarterly model– at the first day in each quarter. Barring this, there must be some odd relation between the two underlying continuous-time series y and x, other than $\mathrm{d}y/\mathrm{d}t = x/\kappa$; more about this in section 8 below.

3 Continuous time

Let us assume that the data generating process runs in continuous time[1], with continuous-time series $y(t)$ and $x(t)$. Without loss of generality the length of the discrete time period is unity. Assuming the variables are flows, observed discrete-time series Y_t and X_t are defined for integer t as

$$Y_t \equiv \int_t^{t+1} y(\tau)\mathrm{d}\tau \tag{3.1}$$

and

$$X_t \equiv \int_t^{t+1} x(\tau)\mathrm{d}\tau \tag{3.2}$$

respectively. We shall also use the moving sum

$$\Xi(t) \equiv \int_t^{t+1} x(\tau)\mathrm{d}\tau \tag{3.3}$$

defined for real t. For integer t we have $\Xi(t) = X_t$.

In continuous-time econometrics much effort has been put into the mathematics of the error term and of continuous-time stochastics in general. However, throughout this paper any error term is ignored: the systematic part of the model equations is enough to make our point. Notice that in the discussion in the previous section, no error term is involved.

The time pattern of the exogenous variable in continuous time is relevant here, as is clear from the discussion at the end of the previous section.

Finally notice that if Y_t were the capital stock, dated, say, at the beginning of the time periods, we would simply have $Y_t \equiv y(t)$ instead of definition (3.1). Then the term with one half in equation (2.1) must be removed. However, in this paper only relations between flows are studied. These relations might be any kind of behavioral relation; not only book-keeping relations such as equation (2.1).

[1] Alternatively, one might assume a data generating process in discrete time, with a higher frequency than the observed data. See e.g. (Maddala, 1977, pp. 374-376) and the references cited there. This give roughly the same result as we will find in this paper: estimate a “geometrically declining lag distribution leaving the first coefficient free”.

4 Sims' distributed lags

The basic reference equation of this paper is the partial adjustment model (1.1). The speed of adjustment to a change in X_t is determined by ρ. Stability requires $-1 < \rho < 1$. Special attention will be paid to the limiting case $\rho \to 1$.

The inclusion of the exogenous X_t is essential here. Otherwise we would have a "pure" time series model, such as Chambers (1999) and Teles and Wei (2000), and most of the modern continuous-time finance models.

An alternative representation of equation (1.1), with an exponentially distributed lag, is as follows:

$$Y_t = \phi \sum_{k=0}^{\infty} \rho^k X_{t-k} \tag{4.1}$$

The relation between equations (1.1) and (4.1) is known in econometrics as the Koyck transform; see Koyck (1954). In this paper we will repeatedly switch between these two equivalent forms.

In Sims (1971) distributed lags are discussed. As in the intuitive investment example above, Sims assumed that the actual process takes place in continuous time. Since in practice one works with discrete-time data, he proposed several discrete-time approximations to the underlying continuous-time distributed lag. The coefficients of these discrete-time approximations are a moving average over the assumed underlying continuous lag distribution[2].

In this case we assume of course that the underlying continuous-time lag distribution is exponential:

$$y(t) = \phi_c \int_0^{\infty} \rho^s x(t-s) \mathrm{d}s \tag{4.2}$$

with some ϕ_c. The relation between $y(t)$ and Y_t (and between $x(t)$ and X_t) was discussed in section 3 above. Since s is not restricted to integers, ρ must be positive, resulting in $0 < \rho < 1$.

In our Appendix A, one of Sims' discrete-time approximations is applied to continuous-time lag distribution (4.2), and it is shown that indeed this produces an exponential discrete-time lag distribution, with exponent ρ. This result is transformed back to the autoregressive form, giving:

$$Y_t = \rho Y_{t-1} + \phi (X_t + r X_{t-1}) \tag{4.3}$$

with

$$r \equiv \frac{1-\rho+\rho \log \rho}{\rho - 1 - \log \rho} \tag{4.4}$$

and with ϕ proportional[3] to ϕ_c.

[2] In Ten Cate (1993) this is applied to polynomial lag distributions (Almon lag distributions).

[3] Note that the relation between ϕ and ϕ_c is not relevant here, since our purpose is to find a good discrete-time model, and not to express a discrete-time model in terms of continuous-time parameters. Also, in the discrete-time equations in this paper the parameters ϕ and ρ are not labeled to distinguish between their different continuous-time origin.

Hence it is better to use equation (4.3) instead of the naive equation (1.1) if one assumes that the actual process takes place in continuous time. Note that in addition to the lagged Y_t, the proposed equation (4.3) also has a lagged X_t in the equation. This does not increase the number of parameters, however, since r is a known function of ρ.

With $0 < \rho < 1$ we have $0 < r < 1$. More specifically, for small ρ we have a small r, and for large ρ we have:

$$\lim_{\rho \to 1} r = 1 \tag{4.5}$$

Hence for $\rho \to 1$:

$$Y_t = Y_{t-1} + \phi(X_t + X_{t-1}) \tag{4.6}$$

Integrated over time:

$$Y_t = 2\phi \left(\frac{1}{2} X_t + X_{t-1} + X_{t-2} + \cdots \right) \tag{4.7}$$

This is the form of the relation between production capacity and investment in equation (2.1), with $1/\kappa = 2\phi$.

As discussed in Ten Cate (1993), equation (4.3) above, and hence also equations (4.6) and (4.7), are exact if $\Xi(t)$, defined in (3.3), is piece-wise linear, i.e. linear within the discrete time periods. This amounts to the original series $x(t)$ having an arbitrary additive seasonal pattern; this is slightly more general than $x(t)$ being constant within the time periods[4].

[4] This might seem to contradict the intuition developed in section 2 above, with the investment example: doesn't it matter whether this seasonal pattern has its highest value early or late within the period? Note that early investments contribute more to the production capacity than late investments. To solve this puzzle, assume that this seasonal pattern consists of the investment of one machine every year, at the same date in each year. Now let us shift this date backwards in time – say one month. Then indeed in every year the production capacity Y_t increases with one month's production capacity of this machine. However, this has no effect on the difference $Y_t - Y_{t-1}$, and hence it leaves equation (4.6) unchanged. Of course equation (4.3) is more complicated; here is no effect on $Y_t - \rho Y_{t-1}$.

5 Bergstrom's approximate analog

In this section and the next two sections, we start from the partial adjustment equation (1.1), written as a difference equation:

$$\Delta Y_t = \phi X_t - (1-\rho)Y_{t-1} \tag{5.1}$$

with the difference Δ defined by $\Delta Y_t \equiv Y_t - Y_{t-1}$. One might assume then that the data are actually generated in continuous time by a differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \phi_c x(t) - \gamma y(t) \tag{5.2}$$

with $\gamma > 0$. With

$$\gamma = -\log\rho \tag{5.3}$$

the differential equation (5.2) is equivalent with the continuous-time distributed lag equation (4.2); they are related by the continuous-time Koyck transformation[5].

Bergstrom (1966) studied difference equations as discrete-time analogs of differential equations. This has resulted in a lot of follow-up research; see for instance the review in Bergstrom (1993).

Bergstrom's approximate discrete-time analog of equation (5.2) is defined by replacing the differential by the difference and the levels by averages over time:

$$\Delta Y_t = \phi_c \frac{X_t + X_{t-1}}{2} - \gamma \frac{Y_t + Y_{t-1}}{2} \tag{5.4}$$

or, with the left-hand side written as in our basic partial adjustment equation (1.1):

$$Y_t = \rho_a Y_{t-1} + \phi(X_t + X_{t-1}) \tag{5.5}$$

with ϕ proportional to ϕ_c and with the ρ_a (where the a indicates 'approximate'):

$$\rho_a = \frac{2-\gamma}{2+\gamma} \tag{5.6}$$

which is smaller than $\rho = \exp(-\gamma)$ as in (5.3) above. This is especially relevant in the continuous-time literature, where the purpose is to translate an estimate of ρ back to the underlying continuous-time γ parameter[6]. However, for the present analysis it matters only that

[5] See for instance (Wymer, 1993, p.39, note 1). In engineering language it is said that the system (5.2) is described by the distributed lag equation (4.2) using the convolution of the impulse response function $h(t) = \rho^t$ with the input $x(t)$; see for instance Oppenheim and Willsky (1983), equation (3.54) at p.98, or Sinha (1991), equation (2.86) at p.74.

[6] It has been overlooked in the econometrical continuous-time literature that $\gamma > 2$ is equivalent to a negative ρ_a in (5.6) and hence an oscillating Y_t in (5.5), which is incompatible with an assumed underlying first-order continuous-time system. Hence the estimated mean lag $1/\gamma$ in model (5.2) can not be smaller than one half of the unit time period. In other words, this is not applicable to 'fast' models – or seemingly fast models, in fact. This limitation holds also for higher-order models. See Ten Cate (2002), where it is also shown that this method is the same as the Tustin transformation, or Bilinear transformation, in engineering.

there is no assumption about the time pattern of $x(t)$, or $\Xi(t)$, for which there is an equation of the form of (5.5) with arbitrary ρ that holds exactly if the underlying continuous-time model is equation (5.2), since $\rho_a \neq \rho$ for any $\gamma > 0$. Hence its name: the approximate analog.

In summary: as in the previous section, taking account of the continuous-time nature of the underlying process introduces a term with the lagged X_t in the right-hand side of (5.5). Compare with the result (4.3) in the previous section: lagged and unlagged X_t always have the same coefficient here.

6 Bergstrom's exact quadratic analog

Bergstrom's exact discrete-time analog of equation (5.2) is derived from solving for $y(t)$ over time, starting from, say, $y(0)$, and computing the discrete-time Y_t from this solution. The continuous-time series $x(t)$ does not appear in the result. This is achieved by using an assumption about its time pattern: a quadratic function of time. Appendix C shows this for a slightly different case, discussed in the next section. Here we have:

$$Y_t = \rho Y_{t-1} + \phi \left(X_t + \frac{r_1}{r_0} X_{t-1} + \frac{r_2}{r_0} X_{t-2} \right) \tag{6.1}$$

with

$$r_0 \equiv \left(-\gamma^{-3} + \frac{1}{2}\gamma^{-2} \right) \rho + \gamma^{-3} - \frac{3}{2}\gamma^{-2} + \gamma^{-1} \tag{6.2}$$

$$r_1 \equiv \left(2\gamma^{-3} - \gamma^{-1} \right) \rho - 2\gamma^{-3} + 2\gamma^{-2} \tag{6.3}$$

$$r_2 \equiv \left(-\gamma^{-3} - \frac{1}{2}\gamma^{-2} \right) \rho + \gamma^{-3} - \frac{1}{2}\gamma^{-2} \tag{6.4}$$

and with γ defined according to (5.3) and with ϕ proportional to ϕ_c. See for instance (Wymer, 1993, p. 49).

With flows this is exact in the sense that if $\Xi(t)$, defined in (3.3), is quadratic then (6.1) satisfies the differential equation (5.2) exactly. (With stocks, the series $x(t)$ itself must be quadratic.) Since the series must be quadratic over each pair of adjacent time periods, it must be quadratic over the *entire* time range, thus excluding business cycles.

With $\gamma \to 0$ we have $\rho \to 1$, the case discussed several times above, related to the intuitive example in section 2. Using the third order approximation $\rho = \exp(-\gamma) \approx 1 - \gamma + \gamma^2/2 - \gamma^3/6$ it is easily shown that equation (6.1) becomes in this case:

$$\Delta Y_t = \phi \left(X_t + \frac{8}{5} X_{t-1} - \frac{1}{5} X_{t-2} \right) \tag{6.5}$$

Integrated over time we have:

$$Y_t = \frac{12}{5} \phi \left(\frac{5}{12} X_t + \frac{13}{12} X_{t-1} + X_{t-2} + X_{t-3} + \cdots \right) \tag{6.6}$$

This differs slightly from Sims' result in equation (4.7) above.

7 Bergstrom's exact linear analog

Although traditionally Bergstrom's exact analog is used with a quadratic assumption as in the previous section, it can also be used with a linear assumption. Let $\Xi(t)$ be piecewise linear, as with Sims' approach, discussed in section 4 above. In Appendix C it is shown that then equation (6.1) simplifies to

$$Y_t = \rho Y_{t-1} + \phi \left(X_t + \frac{r_1^*}{r_0^*} X_{t-1} \right) \tag{7.1}$$

with

$$r_0^* \equiv \gamma^{-2}\rho - \gamma^{-2} + \gamma^{-1} \tag{7.2}$$

$$r_1^* \equiv -\left(\gamma^{-2} + \gamma^{-1}\right)\rho + \gamma^{-2} \tag{7.3}$$

$$\phi \equiv r_0^* \phi_c \tag{7.4}$$

It is easily seen that this can be written as Sims' equations (4.3) and (4.4) above:

$$r = \frac{r_1^*}{r_0^*} = \frac{-\left(\gamma^{-2} + \gamma^{-1}\right)\rho + \gamma^{-2}}{\gamma^{-2}\rho - \gamma^{-2} + \gamma^{-1}} = \frac{-(1+\gamma)\rho + 1}{\rho - 1 + \gamma} = \frac{1 - \rho + \rho \log \rho}{\rho - 1 - \log \rho} \tag{7.5}$$

using relation (5.3). This should not come as a surprise, since the two methods are based on equivalent models and they require the same time pattern of $\Xi(t)$ to be exact.

8 The inverse approximate analog

As a step aside from the main theme of this paper, it might be interesting to see what continuous-time model is implied if the extra term with X_{t-1} in the various difference equations above is not included.

For this purpose the inverse of Bergstrom's approximate analog is derived in Appendix B, using the engineering literature. This inverse analog is found by replacing the discrete-time lag operator by the continuous-time operator $(2-\mathrm{d}/\mathrm{d}t)/(2+\mathrm{d}/\mathrm{d}t)$. This is applied to the "naive" discrete-time model equation (5.1), or (1.1), showing that this model implies at the continuous time level:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \phi_c x(t) - \gamma y(t) + \frac{1}{2}\phi_c \frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{8.1}$$

This differs from the continuous-time model (5.2) by the term with $\mathrm{d}x(t)/\mathrm{d}t$. Here the adjustment to a shock in $x(t)$ is no longer the exponential path defined by γ. In fact the very notion of a shock in $x(t)$ is a problem here, since at the moment of the shock the derivative $\mathrm{d}x(t)/\mathrm{d}t$ does not exists.

Substituting $\gamma = 0$ and $\phi_c = 1/\kappa$ into equation (8.1) gives us the "odd" relation mentioned at the end of section 2:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{1}{\kappa}\left(x(t) + \frac{1}{2}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) \tag{8.2}$$

Here the production capacity increases not only with investment, but also with the increase of investment! It is easily seen that Bergstrom's approximate analog of this equation is indeed the naive

$$\Delta Y_t = \frac{1}{\kappa} X_t \tag{8.3}$$

without X_{t-1}, or

$$Y_t = \frac{1}{\kappa}\left(X_t + X_{t-1} + X_{t-2} + \cdots\right) \tag{8.4}$$

without the one half which occurs in equation (2.1).

9 Conclusion

Most economic processes run more or less continuously in time, from day to day, and not with the frequency of the empirical data such as yearly or quarterly or monthly. The proposition of this paper is that nevertheless modeling in the low frequency of the data makes sense, if the proper dynamic specification is chosen. First, this has been discussed using a simple intuitive example. Next, the continuous-time econometric literature has been used to demonstrate this more formally. It has been shown that the continuous-time models of Sims and Bergstrom are closely related. The inverse of Bergstrom's approximate analog has been introduced and used.

The result is a simple suggestion: a partial adjustment model such as (1.1) might be refined by adding a term with X_{t-1} to the right-hand side. The specific form of this term varies with the approach chosen, as follows.

- Bergstrom's approximate discrete-time analog implies that this term is simply equal to

$$\phi X_{t-1} \tag{9.1}$$

  where ϕ is the coefficient of the unlagged X_t. See equation (5.5) above.
- Based on Sims' work, it is suggested to multiply (9.1) with a specific function of ρ. See equation (4.3) above. For $\rho \to 1$ this function tends to unity, reducing the term to (9.1). See equation (4.6) above.

  The linear form of Bergstrom's exact discrete-time analog gives the same result. See equation (7.1) above.
- Bergstrom's standard quadratic exact discrete-time analog also suggests such a function. This function tends to 1.6 for $\rho \to 1$. Also a term with X_{t-2} is included here. See equation (6.5) above.

Apart from empirical evidence of course, I tend to prefer the first two methods, which can reproduce the intuitive example in section 2. The third method cannot do this.

Appendix A The equations (4.3) and (4.4)

A.1 Sims' approach

In this section we summarize Sims' approach, following Sims (1971). Consider the following distributed-lag equation, with lag operator L defined by $\mathrm{L}^k X_t \equiv X_{t-k}$:

$$Y_t = \left(\sum_{k=0}^{\infty} b_k \mathrm{L}^k\right) X_t \tag{A.1}$$

In the case of a finite length lag distribution, we have $b_k = 0$ beyond the lag length.

The general form of the underlying continuous-time distributed lag model is:

$$y(t) = \int_0^{\infty} \beta(s) x(t-s) \mathrm{d}s \tag{A.2}$$

In general it is not possible to obtain the discrete-time lag equation (A.1) by substituting the continuous-time lag equation (A.2) into definition (3.1): one does not get rid of the continuous-time $x(t)$. Sims (1971) suggested several methods to solve this problem based on assumptions about $x(t)$. One of these methods, based on (Sims, 1971, Proposition A) gives the coefficients b_k in (A.1) as a weighted moving average over β:

$$\begin{aligned} b_k &= \int_{-1}^{1} (1-|s|)\beta(k+s)\mathrm{d}s \\ &= \int_{-1}^{0} (1+s)\beta(k+s)\mathrm{d}s + \int_{0}^{1} (1-s)\beta(k+s)\mathrm{d}s \end{aligned} \tag{A.3}$$

for $k = 1, 2, 3, \ldots$ The weighting function has the form of a 'tent': the function $1 - |s|$ with $-1 \leq s \leq 1$. This local weighting smoothes any sharp peak in the continuous-time lag distribution. In a sense, observing in discrete time is like being unable to see sharp outlines while looking through fog.

It is assumed that $\beta(s)$ vanishes for negative s, as indeed in equation (A.2). This is equivalent to y being dependent only on the past and present of x, and not on the future of x. Then for $k = 0$ we have:

$$b_0 = \int_0^1 (1-s)\beta(s)\mathrm{d}s \tag{A.4}$$

Thus, b_0 misses the left half of the tent, due to $\beta(s)$ being zero for negative s. Notice that if β is constant at the near end, then b_0 is one half of b_1, as in the intuitive example in section 2.

See the end of section 4 above for a discussion of the cases in which this approach is exact.

A.2 Sims' approach applied to the exponential lag distribution

Let the continuous-time lag distribution be as in model (4.2):

$$\beta(s) = \phi_c \rho^s \tag{A.5}$$

The approximate discrete-time lag distribution based on Sims (1971) is found by substituting (A.5) into equation (A.3). Then for $k = 1, 2, 3, \ldots$ we have:

$$\begin{aligned} b_k &= \phi_c \int_{-1}^{0} (1+s)\rho^{k+s} \mathrm{d}s + \phi_c \int_{0}^{1} (1-s)\rho^{k+s} \mathrm{d}s \\ &= \frac{\phi_c}{\log\rho} \left[\left(1 + s - \frac{1}{\log\rho} \right) \rho^{k+s} \right]_{s=-1}^{0} \\ &\quad + \frac{\phi_c}{\log\rho} \left[\left(1 - s + \frac{1}{\log\rho} \right) \rho^{k+s} \right]_{s=0}^{1} \\ &= \frac{\phi_c}{\log\rho} \left(\left(1 - \frac{1}{\log\rho} \right) \rho^k + \frac{1}{\log\rho} \rho^{k-1} \right. \\ &\quad \left. + \frac{1}{\log\rho} \rho^{k+1} - \left(1 + \frac{1}{\log\rho} \right) \rho^k \right) \\ &= \frac{\phi_c}{(\log\rho)^2} \rho^k \left(\rho - 2 + \frac{1}{\rho} \right) \\ &= a\rho^k \end{aligned} \tag{A.6}$$

with

$$a \equiv \frac{\phi_c}{(\log\rho)^2} \left(\rho - 2 + \frac{1}{\rho} \right) \tag{A.7}$$

Hence the approximate discrete-time lag distribution is indeed exponential as well, with the same decay rate parameter ρ as in the assumed underlying continuous-time lag distribution.

In order to find the current period coefficient b_0, assumption (A.5) is also substituted into equation (A.4):

$$\begin{aligned} b_0 &= \phi_c \int_{0}^{1} (1-s)\rho^{s} \mathrm{d}s \\ &= \frac{\phi_c}{\log\rho} \left[\left(1 - s + \frac{1}{\log\rho} \right) \rho^{s} \right]_{s=0}^{1} \\ &= \phi_c \left(\rho - 1 - \log\rho \right) / \left(\log\rho \right)^2 \end{aligned} \tag{A.8}$$

Thus at the discrete-time level we have the exponential lag distribution $b_k = a\rho^k$; however with the current period coefficient b_0 not equal to a. The factor b_0/a is a monotonously increasing function of ρ. With $0 < \rho < 1$ we have $0 < b_0/a < \frac{1}{2}$. More precisely:

$$\lim_{\rho \to 1} \frac{b_0}{a} = \frac{1}{2} \tag{A.9}$$

Here the lag distribution β is near constant and the current-period factor b_0/a is near one half. Compare with the investment example of section 2, where it is one half.

For smaller ρ (faster rate of decay) the current-period factor b_0/a is even smaller. To give a specific example: if $\rho = 0.5$ then $b_0/a = 0.39$; in that case the lag distribution is not equal to

$$1 \;\; 0.5 \;\; 0.25 \;\; 0.125 \;\; \cdots \tag{A.10}$$

but equal to

$$0.39 \;\; 0.5 \;\; 0.25 \;\; 0.125 \;\; \cdots \tag{A.11}$$

The sharp peak at the near end of the lag distribution is indeed smoothed as noted after equation (A.3) above. If (A.11) is normalized to add up to unity then it resembles quite closely Fig. 6.6b in (Hendry, 1995, p. 217)); with $\rho = 0.55$ the two distributions are very close[7].

The results (A.6) through (A.8) can be transformed back to the autoregressive form, with the lagged Y_t, as follows. The equations for the coefficients in (A.6) and (A.8) are substituted into the lag distribution of (A.1):

$$\begin{aligned} Y_t &= \left(\sum_{k=0}^{\infty} b_k \mathrm{L}^k \right) X_t \\ &= \left(b_0 + a \sum_{k=1}^{\infty} \rho^k \mathrm{L}^k \right) X_t \\ &= \left(b_0 + \frac{a\rho \mathrm{L}}{1-\rho \mathrm{L}} \right) X_t \end{aligned} \tag{A.12}$$

Hence:

$$\begin{aligned} (1-\rho\mathrm{L})Y_t &= \left(b_0\left(1-\rho\mathrm{L}\right) + a\rho\mathrm{L}\right) X_t \\ &= \left(b_0 + \left(a-b_0\right)\rho\mathrm{L}\right) X_t \\ &= (1+r\mathrm{L}) b_0 X_t \\ &= (1+r\mathrm{L})\phi X_t \end{aligned} \tag{A.13}$$

with:

$$r \equiv \frac{(a-b_0)\rho}{b_0} = \frac{1-\rho+\rho\log\rho}{\rho-1-\log\rho} \tag{A.14}$$

and

$$\phi \equiv b_0 = \phi_c \left(\rho - 1 - \log\rho\right) / (\log\rho)^2 \tag{A.15}$$

[7] I got the idea to write this paper when I saw this lag distribution in Hendry’s book.

Appendix B The equation (8.1)

In order to derive equation (8.1), the inverse of Bergstrom's approximate analog is introduced, for the linear case. As discussed in section 5 above, Bergstrom's analog is obtained by replacing, for an arbitrary series $y(t)$, the differential quotient $\mathrm{d}y(t)/\mathrm{d}t$ by the difference $\Delta Y_t \equiv Y_t - Y_{t-1}$, and the level $y(t)$ by the mean $(Y_t + Y_{t-1})/2$. It is assumed without loss of generality that the discrete time unit is of length one.

For linear models, these two operations can be combined by replacing the differential quotient by the difference divided by the mean:

$$\mathrm{D} = \frac{\Delta}{\frac{1}{2}(1+\mathrm{L})} \tag{B.1}$$

or

$$\mathrm{D} = 2\frac{1-\mathrm{L}}{1+\mathrm{L}} \tag{B.2}$$

where D is the differential operator $\mathrm{d}/\mathrm{d}t$ and as before the lag operator L is defined by $\mathrm{L}Y_t \equiv Y_{t-1}$. It is easily verified that in this way differential equation (5.2) is indeed transformed into equation (5.5). (Notice that in this way also Bergstrom's approximate analog of higher order equations can be found without first decomposing them into first order equations.)

In the engineering literature of dynamic systems this transformation is called the Bilinear Transformation or the Tustin transformation, and written as:

$$s = 2\frac{z-1}{z+1} \tag{B.3}$$

where s and z are the Laplace transform variable and the z-transform variable respectively. In the context of linear models, s coincides with D and z^{-1} coincides with L. See Ten Cate (2002), where the hitherto overlooked relation between continuous-time econometrics and engineering mathematics is discussed. See also for instance Sinha and Rao (1991) for a discussion of the relation between continuous and discrete time models in engineering language.

The inverse of (B.2) is:

$$\mathrm{L} = \frac{2-\mathrm{D}}{2+\mathrm{D}} \tag{B.4}$$

Note that this is of the same form as relation (5.6). As shown in Ten Cate (2002), this holds more in general: the relation between the two operators is the same as the relation between the roots of the characteristic equations.

Equation (B.4) is applied to difference equation (5.1), or its equivalent (1.1), giving:

$$\left(1-\rho\frac{2-\mathrm{D}}{2+\mathrm{D}}\right)y(t) = \phi x(t) \tag{B.5}$$

This can be rewritten as follows, proving equation (8.1):

$$\mathrm{D}y(t) = \frac{1}{1+\rho}\left(-2(1-\rho)y(t) + 2\phi x(t) + \phi \mathrm{D}x(t)\right)$$

$$= -\gamma y(t) + \frac{1}{1+\rho}\left(2\phi x(t) + \phi \mathrm{D}x(t)\right)$$

$$= -\gamma y(t) + \phi_c x(t) + \frac{1}{2}\phi_c \mathrm{D}x(t) \tag{B.6}$$

using (5.6) and defining:

$$\phi_c \equiv \frac{2}{1+\rho}\phi \tag{B.7}$$

Appendix C The equations (7.1) – (7.4)

To make this paper more self contained, the derivation is given of the linear form of Bergstrom's exact analog. The solution of differential equation (5.2) satisfies for all real τ:

$$\begin{aligned} y(\tau) &= \mathrm{e}^{-\gamma} y(\tau-1) + \int_0^1 \mathrm{e}^{-\gamma s} \phi_c x(\tau-s)\mathrm{d}s \\ &= \rho y(\tau-1) + \phi_c \int_0^1 \rho^s x(\tau-s)\mathrm{d}s \end{aligned} \tag{C.1}$$

Integration over $t \leq \tau \leq t+1$ for integer t gives:

$$\begin{aligned} Y_t &= \rho Y_{t-1} + \phi_c \int_t^{t+1} \int_0^1 \rho^s x(\tau-s)\mathrm{d}s\,\mathrm{d}\tau \\ &= \rho Y_{t-1} + \phi_c Z_t \end{aligned} \tag{C.2}$$

with

$$\begin{aligned} Z_t &\equiv \int_t^{t+1} \int_0^1 \rho^s x(\tau-s)\mathrm{d}s\,\mathrm{d}\tau \\ &= \int_0^1 \rho^s \int_t^{t+1} x(\tau-s)\mathrm{d}\tau\,\mathrm{d}s \\ &= \int_0^1 \rho^s \Xi(t-s)\mathrm{d}s \end{aligned} \tag{C.3}$$

where Ξ is defined in (3.3). Assuming Ξ is piecewise linear we have for integer t and $0 \leq s \leq 1$:

$$\begin{aligned} \Xi(t-s) &= \Xi(t) + s\left(\Xi(t-1) - \Xi(t)\right) \\ &= X_t + s\left(X_{t-1} - X_t\right) \end{aligned} \tag{C.4}$$

Then

$$\begin{aligned} Z_t &= \int_0^1 \rho^s \left(X_t + s\left(X_{t-1} - X_t\right)\right)\mathrm{d}s \\ &= X_t \int_0^1 \rho^s (1-s)\mathrm{d}s + X_{t-1} \int_0^1 \rho^s s\,\mathrm{d}s \\ &= r_0^* X_t + r_1^* X_{t-1} \end{aligned} \tag{C.5}$$

with:

$$\begin{aligned} r_0^* &\equiv \int_0^1 \rho^s (1-s)\mathrm{d}s \\ &= \left[\left(-\gamma^{-1} + s\gamma^{-1} + \gamma^{-2}\right)\rho^s\right]_{s=0}^1 \\ &= \gamma^{-2}\rho - \gamma^{-2} + \gamma^{-1} \end{aligned} \tag{C.6}$$

and

$$\begin{aligned} r_1^* &\equiv \int_0^1 \rho^s s\,\mathrm{d}s \\ &= \left[-\left(\gamma^{-2} + s\gamma^{-1}\right)\rho^s\right]_{s=0}^1 \\ &= -\left(\gamma^{-2} + \gamma^{-1}\right)\rho + \gamma^{-2} \end{aligned} \tag{C.7}$$

Then

$$
\begin{aligned}
(1-\rho \mathrm{L})Y_t &= \phi_c Z_t \\
&= \phi_c (r_0^* + r_1^* \mathrm{L}) X_t \\
&= \phi (1 + \frac{r_1^*}{r_0^*} \mathrm{L}) X_t
\end{aligned} \tag{C.8}
$$

with

$$
\phi \equiv r_0^* \phi_c \tag{C.9}
$$

As in the other Appendices, use has been made of the lag operator L defined by $\mathrm{L}^k Y_t \equiv Y_{t-k}$.

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