CPB Memorandum

CPB Netherlands Bureau for Economic Policy Analysis

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Number	:	223
Date	:	April 14, 2009

Arithmetic and geometric mean rates of return in discrete time

This memorandum presents some basic equalities and inequalities about rates of return in discrete time, without auto-correlation. The arithmetic and geometric means are discussed. Estimation of the expected payout and the median payout is discussed, including maximum likelihood estimation.

Contents

1 Introduction

This memorandum[1] presents some basic equalities and inequalities about rates of return in discrete time, without auto-correlation.

Modelling stochastic rates of return in discrete time might be simpler than using the concept of continuous time Brownian motion. On the other hand, many results below are only

[1] The author thanks Pascal Janssen (PGGM) and Gijsbert Zwart and Ed Westerhout (CPB) for their help.

approximations, using assumptions such as the one-period return and variance being both much smaller than one. In both approaches numerical results must be computed with small time steps.

First a general model is discussed, requiring only an expected return which is constant over time. Starting in section 4, the case of lognormally distributed returns is discussed, including maximum likelihood estimation and the median payout.

Although most if not all results are not new, this overview might be useful. Your author would have saved time if it were available when he needed it. Comments are invited.

2 The general model

Let S_t be the value of stocks at time t, with:

$$S_t = (1+r_t)S_{t-1} \tag{2.1}$$

for all $t = 1,\ldots,T$. Let S_0 be a given positive number. The r_t are stochastic; they are independently distributed. The form of the distribution is not specified. Assuming limited liability implies for all t:

$$\Pr(r_t < -1) = 0 \tag{2.2}$$

The S_t are independent of the previous series $S_{t-1},\ldots,S_1$. Also, the r_t have the same expectation $\mathrm{E}[r_t]$, denoted by m:

$$\mathrm{E}[r_t] = m \tag{2.3}$$

for all t. Since the r_t and the S_{t-1} are independently distributed, we have for some T:

$$\mathrm{E}[S_T] = \mathrm{E}[(1+r_T)S_{T-1}] = \mathrm{E}[1+r_T]\,\mathrm{E}[S_{T-1}] = (1+m)\mathrm{E}[S_{T-1}] \tag{2.4}$$

Repeating this all the way down to S_0 gives:

$$\mathrm{E}[S_T] = (1+m)^T S_0 \tag{2.5}$$

or

$$(\mathrm{E}[S_T/S_0])^{1/T} - 1 = m \tag{2.6}$$

The arithmetic mean is

$$\widehat{m}_\mathrm{A} \equiv \frac{1}{T}\sum_t r_t = \frac{1}{T}\sum_t \left(\frac{S_t}{S_{t-1}} - 1\right) \tag{2.7}$$

Of course this is an unbiased estimator of m, due to equation (2.3) above:

$$\mathrm{E}[\widehat{m}_\mathrm{A}] = m \tag{2.8}$$

See section 7 below for the maximum likelihood estimation of m under the assumption of lognormal returns.

Let $1+\widehat{m}_{\mathrm{G}}$ be the geometric mean of the $1+r_t$:

$$1+\widehat{m}_{\mathrm{G}} \equiv \left(\prod_t 1+r_t\right)^{1/T} = \left(\frac{S_T}{S_0}\right)^{1/T} \tag{2.9}$$

Hence S_T/S_0 is equal to one plus the geometric mean, to the power T. However, its expected value is equal to one plus the expected value of the *arithmetic* mean, to the power T.

A geometric mean of non-negative numbers is smaller than (or equal to) the arithmetic mean. (For example, with $r_t = \pm 1/2$ the arithmetic mean of $1+r_t$ is 1 and the geometric mean is $\sqrt{0.75} = 0.87$.) Hence we have $\widehat{m}_{\mathrm{G}} \leq \widehat{m}_{\mathrm{A}}$ for every realization of the series $1+r_t$, and hence, with (2.8):

$$\mathrm{E}\left[\widehat{m}_{\mathrm{G}}\right] < m \tag{2.10}$$

Nonzero variance of the r_t is assumed here, giving strict inequality in (2.10).

3 Example

There are several small numerical examples in the literature which illustrate the previous section. For instance the well-written Exhibit 10.6 in the section “Geometric versus arithmetic average” of McKinsey & Company et al. (2000).

A very simple example is the case of $T = 2$ with $r_t = \pm 1/2$, like the example before equation (2.10). With equal probabilities this gives $m = 0$. With $S_0 = 1$, the four possible values of S_T are 0.25, 0.75 (twice) and 2.25. They have equal probability and hence $\mathrm{E}\left[S_T\right] = 1$, which agrees with equation (2.5) above.

The geometric mean return $\widehat{m}_{\mathrm{G}}$, defined in equation (2.9) above, can take on the four values $\sqrt{0.25}-1$, $\sqrt{0.75}-1$ (twice) and $\sqrt{2.25}-1$, giving $\mathrm{E}[\widehat{m}_{\mathrm{G}}] = -0.07$ which is smaller than $m = 0$. This agrees with equation (2.10) above.

Note that after one high and one low r_t (in either order) we get a $\sqrt{S_2/S_0}$ equal to the geometric mean computed in the example before equation (2.10), smaller than $1+m$. This is because the other two possibilities are omitted: twice high and twice low.

4 The lognormal model

Additionally to the assumptions in section 2, it is assumed that all r_t are identically distributed. Moreover, let the distribution of the $1+r_t$ be lognormal:

$$\log(1+r_t) \sim \mathrm{N}\left(\mu, \sigma^2\right) \tag{4.1}$$

for all t. The limited-liability restriction (2.2) above now becomes $\Pr(r_t \leq -1) = 0$.

Also:

$$\log(S_T/S_0) \sim \mathrm{N}\left(T\mu, T\sigma^2\right) \tag{4.2}$$

$$\log(1+\widehat{m}_{\mathrm{G}}) \sim \mathrm{N}\left(\mu, \frac{1}{T}\sigma^2\right) \tag{4.3}$$

It follows from the formulas for the expectation and variance of a lognormal variable that:

$$\mathrm{E}\left[1+r_t\right] = \exp\left(\mu + \frac{1}{2}\sigma^2\right) \tag{4.4}$$

$$\mathrm{E}\left[1+\widehat{m}_{\mathrm{G}}\right] = \exp\left(\mu + \frac{1}{2T}\sigma^2\right) \tag{4.5}$$

$$\mathrm{Var}\left[r_t\right] = (1+m)^2\left(\exp\left(\sigma^2\right) - 1\right) \approx (1+m)^2\sigma^2 \approx \sigma^2 \tag{4.6}$$

where the two $\approx$ signs are associated with small σ^2 and small m, respectively.

5 The expected geometric mean return

An approximation of the difference between the two sides of inequality (2.10) is derived, with lognormal $1+r_t$. With (4.5) above, we have

$$\begin{aligned}\mathrm{E}\left[1+\widehat{m}_{\mathrm{G}}\right] &= \exp\left(\mu + \frac{1}{2T}\sigma^2\right) = \exp\left(\mu + \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2 + \frac{1}{2T}\sigma^2\right) \\ &= (1+m)\exp\left(-\frac{1}{2}\sigma^2 + \frac{1}{2T}\sigma^2\right)\end{aligned} \tag{5.1}$$

Hence with large T we have:

$$\frac{\mathrm{E}\left[1+\widehat{m}_{\mathrm{G}}\right]}{1+m} \approx \exp\left(-\frac{1}{2}\sigma^2\right) < 1 \tag{5.2}$$

Then we have also:

$$\begin{aligned}\mathrm{E}\left[\widehat{m}_{\mathrm{G}}\right] &\approx (1+m)\exp\left(-\frac{1}{2}\sigma^2\right) - 1 \\ &\approx (1+m)\left(1 - \frac{1}{2}\sigma^2\right) - 1 = m - \frac{1}{2}\sigma^2 - m\frac{1}{2}\sigma^2 \\ &\approx m - \frac{1}{2}\sigma^2 \\ &= \mathrm{E}\left[\widehat{m}_{\mathrm{A}}\right] - \frac{1}{2}\sigma^2\end{aligned} \tag{5.3}$$

where the last two $\approx$ signs are associated with small σ^2 and small m, respectively. Compare with (2.10).

6 The arithmetic and geometric returns

The result of the previous section suggest the following two concepts, used in communication between practitioners:

$$\text{arithmetic rate of return} \equiv m \tag{6.1}$$

$$\text{geometric rate of return} \equiv m - \frac{1}{2}\mathrm{Var}\left[r_t\right] \approx m - \frac{1}{2}\sigma^2 \tag{6.2}$$

Equations (4.6) and (5.3) are used for (6.2).

7 Maximum likelihood estimation of m and $\mathrm{E}\left[S_T\right]$

In section 2 above, the unbiased estimation of $m = \mathrm{E}\left[r_t\right]$ was discussed. The maximum likelihood estimate of the same is, using equation (4.4) above:

$$\widehat{m}_{\mathrm{ML}} = \exp\left(\widehat{\mu} + \frac{1}{2}\widehat{\sigma}^2\right) - 1 \tag{7.1}$$

where $\widehat{\mu}$ and $\widehat{\sigma}$ are the maximum likelihood estimate of μ and σ, respectively. These are the sample mean and standard deviation of the normally distributed $\log(1+r_t)$, respectively. (See any econometrics or statistics textbook.) Then, with equation (2.9) above, we have:

$$\widehat{\mu} = \log(1+\widehat{m}_{\mathrm{G}}) \tag{7.2}$$

and hence with equation (7.1):

$$\frac{1+\widehat{m}_{\mathrm{G}}}{1+\widehat{m}_{\mathrm{ML}}} = \frac{\exp\left(\widehat{\mu}\right)}{\exp\left(\widehat{\mu} + \frac{1}{2}\widehat{\sigma}^2\right)} = \exp\left(-\frac{1}{2}\widehat{\sigma}^2\right) < 1 \tag{7.3}$$

Compare with equation (5.2) above. For large T the standard error of $\widehat{\sigma}^2$ goes to zero and we have $\widehat{\sigma}^2 \approx \sigma^2$; compare Campbell et al. (1997), equation (9.3.31)[2].

The maximum likelihood estimate of $\mathrm{E}\left[S_T\right]$ is $(1+\widehat{m}_{\mathrm{ML}})^T S_0$, using (2.5). In practice of course the $\widehat{m}_{\mathrm{ML}}$ is computed over a historical time range before $t = 0$.

8 The median

The median of a variable is equal to the exp of the median of the log, since both functions are monotonous. With lognormal $1+r_t$, $\log(1+\widehat{m}_{\mathrm{G}})$ is normally distributed; see (4.3) above. Hence

[2] Our T is the n of Campbell et al. (1997) and our r_t is their R_t. We follow them in making no distinction between $\widehat{\sigma}^2$ and $\widehat{\sigma^2}$, because the two are the same, of course. The σ above their equation (9.3.26) must be σ^2.

the median of $\log(1+\widehat{m}_{\mathrm{G}})$ is equal to its expectation. Then

$$\begin{aligned}
\text{Median}\left[S_T/S_0\right] &= \exp\left(\text{Median}\left[\log\left(S_T/S_0\right)\right]\right) = \exp\left(\text{Median}\left[\sum_t \log\left(1+r_t\right)\right]\right) \\
&= \exp\left(\text{Median}\left[T\log\left(1+\widehat{m}_{\mathrm{G}}\right)\right]\right) = \exp\left(T\ \text{Median}\left[\log\left(1+\widehat{m}_{\mathrm{G}}\right)\right]\right) \\
&= \exp\left(T\ \mathrm{E}\left[\log\left(1+\widehat{m}_{\mathrm{G}}\right)\right]\right) \\
&= \exp\left(T\mu\right)
\end{aligned} \tag{8.1}$$

Then

$$\frac{\text{Median}\left[S_T/S_0\right]}{\mathrm{E}\left[S_T/S_0\right]} = \frac{\exp\left(T\mu\right)}{\exp\left(T\left(\mu+\frac{1}{2}\sigma^2\right)\right)} = \rho^T \tag{8.2}$$

with

$$\rho \equiv \exp\left(-\frac{1}{2}\sigma^2\right) < 1 \tag{8.3}$$

Also, with $T \gg \sigma^2/\mu$ we have, using (8.1) and (4.5):

$$\text{Median}\left[\left(S_T/S_0\right)^{1/T}\right] = \left(\text{Median}\left[S_T/S_0\right]\right)^{1/T} = \exp\left(\mu\right) \approx \mathrm{E}\left[1+\widehat{m}_{\mathrm{G}}\right] \tag{8.4}$$

9 When to consider geometric?

9.1 Unbiased estimation of $\mathrm{E}\left[S_T\right]$

The conclusion below equation (2.9) might suggest the use of the arithmetic mean for the estmation of the expected multi-period payout.

However, substituting the arithmetic mean for the m in (2.5) gives a biased estimate of the expected payout, due to the non-linearity of the power raising. Jacquier et al. (2003) find that using a weighted average of the geometric and arithmetic means gives an unbiased estimate of the expected multi-period payout.

The alternative is the maximum likelihood estimator of $\mathrm{E}\left[S_T\right]$, in section 7. (Note that at Campbell et al. (1997), p.367, maximum likelihood estimation of option prices is the only estimation method discussed, without bothering about finite sample bias.)

Finally, note that in simulations where parameter values are assumed, this discussion about estimation is not relevant.

9.2 The median

One might be interested in the median payout rather than the expected payout. Focussing only on a high value the latter may be "courting with ruin" (Samuelson, 1971) if the median payout tends to zero for large T. As can be seen from equation (8.4), for the case of lognormal returns this implies a negative expected geometric mean – or a negative geometric return, from (6.2).

This is also relevant in simulations where parameter values are assumed.

10 Summary of results

With lognormal returns we have:

$$\text{arithmetic rate of return} \equiv \mathrm{E}[r_t] \equiv m = \mathrm{E}[\widehat{m}_{\mathrm{A}}] \approx \mathrm{E}[\widehat{m}_{\mathrm{ML}}] \approx \left(\mathrm{E}\left[S_T/S_0\right]\right)^{1/T} - 1 \tag{10.1}$$

and

$$\begin{aligned}
\text{geometric rate of return} &\equiv m - \frac{1}{2}\mathrm{Var}[r_t] \approx m - \frac{1}{2}\sigma^2 \\
&\approx \mathrm{E}[\widehat{m}_{\mathrm{G}}] = \mathrm{E}\left[\left(S_T/S_0\right)^{1/T}\right] - 1 \\
&\approx \left(\mathrm{Median}\left[S_T/S_0\right]\right)^{1/T} - 1 \\
&= \mathrm{Median}\left[\left(S_T/S_0\right)^{1/T}\right] - 1
\end{aligned} \tag{10.2}$$

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