CPB Memorandum

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A closed-form expression for the optimal capacity of CHP

In this memorandum the optimal capacity of Combined Heat and Power (CHP) is derived, using a simple model with an analytical solution. The solution is expressed as the fraction of the time during which the heat demand exceeds the optimal CHP heat production capacity.

Contents

1 Introduction

Combined Heat and Power (CHP) production can be very efficient, since CHP equipment might have a higher efficiency than electricity from the grid[1].

The economically optimal size of a CHP installation is a variant of the general problem of optimal investment in production capacity with varying demand. On the one hand, the costly capacity must not serve very short-lived demand peaks and operate below full capacity nearly all of the time. On the other hand, the capacity must not be insufficient all the time. There is an optimum between these two extremes.

A numerical example of the model solution is given, with computer code.

This problem has also been studied by Dobbs (1982), using a very detailed analytical model; see our appendix B below. There are also many numerical optimization analyses, such as Lund and Andersen (2005), Ren et al. (2008), Streckiene and Andersen (2008). Conti et al. (2007)

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explore the interesting combination of CHP with heat pumps, with much attention to CO_2 reduction.

2 The model

The heat demand is assumed to be fixed. Let $f(h)$ be the density function of the heat demand h: the heat demand (say, in Watt) is in the range h to $h+\mathrm{d}h$ during a fraction of the time given by $f(h)\mathrm{d}h$. Then the total time is

$$\int_0^\infty f(h)\mathrm{d}h = 1 \tag{2.1}$$

and

$$\int_0^\infty f(h)h\mathrm{d}h \tag{2.2}$$

is the average heat demand over time[2]. The cumulation of the density $f(h)$ is the inverse of the heat demand duration curve.

The heat production capacity of the CHP installation is C. If (and only if) heat demand h exceeds C then heat is produced with a conventional back-up installation.

Then the net financial benefit B of CHP, per unit of time, as a function of C, is:

$$\begin{aligned} B(C) &= b\int_0^\infty f(h)\min(C,h)\mathrm{d}h - kC \\ &= b\left(\int_0^C f(h)h\mathrm{d}h + C\int_C^\infty f(h)\mathrm{d}h\right) - kC \end{aligned} \tag{2.3}$$

The b is the operational revenue of CHP per unit of heat produced and per unit of time (combined, say, per Joule). This b depends on the price of avoided electricity use from the grid and the price of gas and the (technical) efficiency of the various processes. We assume b independent[3] of capacity C.

The k is the capital cost of the investment, per unit of time (the annuity) and per unit of heat production capacity of the CHP. This includes interest payments and depreciation.

This formulation may apply to different settings. For instance with a First Best model for a society as a whole, subsidies and taxes in the computation of b might be treated different from the computation for a private investor. Also, the demand might come from a single house, or from an office block, etcetera.

[2] If $h=0$ during some non-zero fraction of the time then $f(0)$ is not finite. Let $f(0)$ be equal to a constant times the Dirac delta: the derivative of a unit step. Then $\lim_{\varepsilon\to 0}\int_0^\varepsilon f(h)\mathrm{d}h > 0$ and also $\lim_{\varepsilon\to 0}\int_0^\varepsilon f(h)h\mathrm{d}h = 0$. This has no effect on the results.

[3] Alternatively, when CHP is installed in most homes and offices, then the price of electricity might drop during peak hours of heat demand.

3 The solution

The first-order condition for a maximum of the net benefit B is:

$$\frac{\mathrm{d}B(C)}{\mathrm{d}C} = b\left(f(C)C + \int_C^\infty f(h)\mathrm{d}h - Cf(C)\right) - k = b\int_C^\infty f(h)\mathrm{d}h - k = 0 \tag{3.1}$$

Hence the optimal capacity C^* satisfies the following equality between marginal revenue and marginal cost of the investment[4]:

$$b\int_{h=C^*}^\infty f(h)\mathrm{d}h = k \tag{3.2}$$

Or:

$$\int_{h=C^*}^\infty f(h)\mathrm{d}h = \rho \tag{3.3}$$

with ρ being the ratio of capital costs over operational revenue:

$$\rho \equiv k/b \tag{3.4}$$

This can also be written as a closed-form expression:

$$C^* = F(\rho) \tag{3.5}$$

where the function F is the (downward sloping) heat demand duration curve, with

$$F^{-1}(C) \equiv \int_{h=C}^\infty f(h)\mathrm{d}h \tag{3.6}$$

See figure 3.1 below.

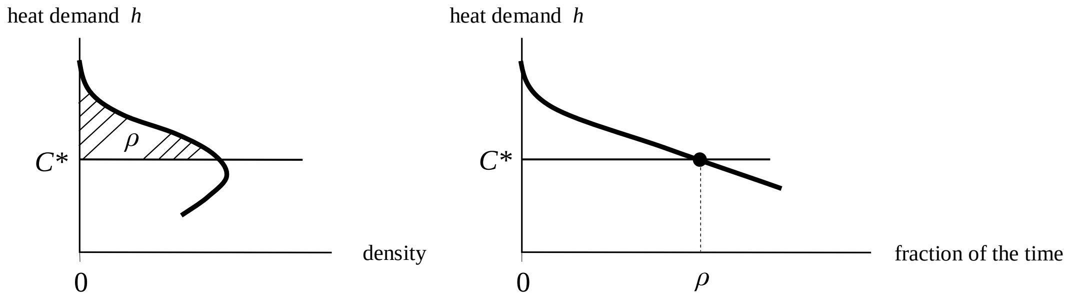
It follows from equation (3.5) that the optimal capacity decreases when ρ increases; that is, when the capital cost k increases or the revenue b decreases, or both. If $k > b$ then CHP is not profitable.

4 Further details of the solution

The optimal capacity C^* might be positive but smaller than the capacity of the minimal available installation; for example with a single house. Then of course it is still optimal to install CHP if the net benefit B in equation (2.3) is positive for this minimal available capacity.

[4] Compare the two sides of equation (3.2) with the marginal revenue curve and the (horizontal) marginal cost curve in Figure 4 of Ren et al. (2008). They use a storage tank (not included in the costs) and hence their marginal revenue function is a “smoothed” version of their (transposed) heat demand duration curve; see our (3.6). (Heat demand data file obtained by courtesy of the authors.)

Figure 3.1 The density curve $f(h)$ in eq. (3.3) and the duration curve F in eq. (3.5)



The optimal net benefit is indeed maximal (not minimal), because the second-order derivative is negative everywhere:

$$\frac{\mathrm{d}^2B(C)}{\mathrm{d}C^2} = \frac{\mathrm{d}}{\mathrm{d}C}\left(b\int_C^\infty f(h)\mathrm{d}h - k\right) = -bf(C) < 0 \tag{4.1}$$

The maximal net benefit is always positive for a positive C^*: substitution of the first-order condition (3.3) into the last member of (2.3) gives

$$B_{\max} = b\int_{h=0}^{C^*} f(h)h\mathrm{d}h > 0 \tag{4.2}$$

This is the operational revenue of the CHP (not counting capital costs) during off-peak hours. (We define off-peak by $h < C$.)

The capital costs are equal to the operational revenue of the CHP during peak hours. This becomes clear when equation (3.2) is multiplied by C^*.

Equation (3.2) is similar to the peak load model for optimal capacity of any capital intensive production for a demand which varies over time, with perfect competition. The electricity market is the prime example. See for instance Ten Cate and Lijesen (2004), section 4.5, "Optimal investment in a nutshell".

5 Formulas for b and k

Formulas for the constants b and k in the solution of the model are straightforward. They are used in the numerical example below. For b, the revenue per unit of heat produced by CHP, we have:

$$b = p_e\frac{\eta_e}{\eta_{heat}} - p_{gas}\left(\frac{1}{\eta_{heat}} - \frac{1}{\eta_{ref}}\right) = p_{gas}\left[\frac{1}{\eta_{ref}} - \frac{1}{\eta_{heat}}\left(1 - \frac{\eta_e}{\eta_{egrid}}\right)\right] \tag{5.1}$$

The p_e and p_{gas} are the price (or unit costs) of electricity from the grid and the price (or unit costs) of gas; both per unit of energy. (The same price p_e applies to both buying and selling from/to the grid.) The η_{heat}, η_e and η_{ref} are the efficiency of heat output of CHP, electrical output of CHP, and (heat) output of the reference system, respectively. The "economic efficiency" of the electricity from the grid (with respect to gas) is:

$$\eta_{\text{egrid}} \equiv \frac{p_{\text{gas}}}{p_{\text{e}}} \tag{5.2}$$

This is related to the "spark spread". We have $\eta_{\text{e}} < \eta_{\text{egrid}} < \eta_{\text{gas}} < \eta_{\text{ref}}$.

For k, the capital costs flow per unit of CHP heat production capacity, we have[5]:

$$k = \frac{\eta_{\text{e}}}{\eta_{\text{heat}}} \frac{Kr}{1 - \exp(-rT)} \tag{5.3}$$

The K is the price of the CHP per unit of e-capacity. (Possibly corrected for the decrease of the required auxiliary conventional heat production capacity, linearly related to the CHP production capacity.) The T is the CHP equipment life time. The r is the discount rate. Note that

$$\lim_{T \to \infty} \frac{Kr}{1 - \exp(-rT)} = Kr \tag{5.4}$$

which leaves only interest. With l'Hôpital's rule:

$$\lim_{r \to 0} \frac{Kr}{1 - \exp(-rT)} = \left. \frac{K}{T \exp(-rT)} \right|_{r=0} = \frac{K}{T} \tag{5.5}$$

which leaves only depreciation.

6 A numerical example

A numerical example of the model is given. A value of $\rho = k/b$ is computed, which gives the value at the horizontal axis of the duration curve, as shown in equation (3.5). See the Octave/Matlab program in Appendix A.

The discount rate r is set at 10% per year. This is well below the discount rate of 19% implied by the cost-recovery period of 5 years over a life time T of 15 years used in De Jong et al. (2008). (See equation (5.3) above: solve $r/(1 - \exp(-15r)) = 1/5$ for r.)

The (additional) investment costs of CHP is set at $K = 1800$ euro/kWe, not including a hot water storage tank. Retail energy prices are used, taken from SenterNovem (2009).

The main result is $\rho = 24\%$; see the bottom line of figure A.2 in Appendix A. Hence the optimal capacity is less than the heat demand during 24% of the time.

This result can be expressed as 88 days per year. Compare with the duration curve[6] of figure 3.4 in De Jong et al. (2008), for a Dutch house. A low-end CHP installation of 1 kWe (or nearly 6 kW-heat) is equal to the height of that duration curve at 50 days. This suggests, that the optimal capacity is smaller.

[5] The $\exp(-rT)$ is the continuous time version of the discrete time expression $(1+r)^{-T}$. Let $r \to 0$ with fixed rT.

[6] There must be some error in the computer programming here, since the duration curve is both downward and upward sloping.

Appendix A Computer program with results

Figure A.1 Computer program for Octave or Matlab

```
efficiency_chp_heat = 0.75
efficiency_chp_electr = 0.15
efficiency_reference = 0.90
price_chp_euro_per_kW_electr = 1800
caloric_content_gas_MJ_per_m3 = 31.7
life_time_chp_years = 15
discount_rate_per_year = 0.10
cost_electr_grid_euro_per_kWh = 0.20
cost_gas_euro_per_m3 = 0.70

disp(" Results:");
capital_costs_euro_per_kW_heat_and_per_year = ...
(efficiency_chp_electr / efficiency_chp_heat) ...
* price_chp_euro_per_kW_electr ...
* discount_rate_per_year ...
/ (1-exp(-discount_rate_per_year*life_time_chp_years))

cost_gas_euro_per_kWh = ...
cost_gas_euro_per_m3 * 3.6 / caloric_content_gas_MJ_per_m3

value_electr = cost_electr_grid_euro_per_kWh ...
* (efficiency_chp_electr / efficiency_chp_heat)

value_extra_gas = cost_gas_euro_per_kWh ...
* (1/efficiency_chp_heat - 1/efficiency_reference)

revenue_euro_per_kWh_heat = value_electr - value_extra_gas

horizontal_axis_duration_curve = ...
(capital_costs_euro_per_kW_heat_and_per_year / (24*365)) ...
/ revenue_euro_per_kWh_heat
```

Figure A.2 Results of the computer program

```
capital_costs_euro_per_kW_heat_and_per_year =  46.340
cost_gas_euro_per_kWh =  0.079495
value_electr =  0.040000
value_extra_gas =  0.017666
revenue_euro_per_kWh_heat =  0.022334
horizontal_axis_duration_curve =  0.23685
```

Appendix B Comparison with Dobbs, 1982

Dobbs (1982) presents an analytic model of optimal CHP. His model is much more general than our model above. It includes a market for heat and a market for electricity, with a time-varying demand function. The model has $3+8n$ free variables, including $5n$ Lagrange multipliers, where n is the number of discrete time periods with fixed given duration. (Our model above has only one free variable, namely C.) Nothing changes during a time period. Because of the complexity of his model, Dobbs studies the result for two periods.

Dobbs' net benefit function W, in his equation (10), contains three terms like kC in our (2.3) above. These are related to three production capacity variables, namely for CHP electricity production, CHP heat production, and conventional heat production, respectively. The first two terms can easily be combined using Dobbs' assumptions about constant efficiencies of the two CHP production processes. (See also the bottom of the right-hand column of Dobbs' page 278.) The last term can also be combined with the others (apart from a constant), assuming a decrease of the required conventional heat production capacity which is linearly related to the CHP production capacity.

Using Dobbs' alternative assumption of a fixed electricity price, the first term of his net benefit function becomes simply the value of the produced electricity. See his page 281, first sentence of the left-hand column.

Finally, with our assumption of a given demand for heat which must be satisfied, the second term in Dobbs' net benefit function becomes a constant and can be omitted.

Then Dobbs' net benefit function W is as follows, with t_i being the length of time period i; the H_i and E_i and F_i are respectively the total heat produced, the electricity produced and the total fuel consumed during time period i. Using his (10) and (11) and $E_i/\eta_{\text{e}} = H_{\text{chp},i}/\eta_{\text{heat}}$ as discussed above, we get:

$$\begin{aligned}
W &= \sum_{i=1}^{n} t_i \left(p_{\text{e}} E_i - p_{\text{gas}} F_i\right) - kC \\
&= \sum_{i=1}^{n} t_i \left(p_{\text{e}} H_{\text{chp},i} \frac{\eta_{\text{e}}}{\eta_{\text{heat}}} - p_{\text{gas}} \left(\frac{H_{\text{chp},i}}{\eta_{\text{heat}}} + \frac{H_i - H_{\text{chp},i}}{\eta_{\text{ref}}}\right)\right) - kC \\
&= \sum_{i=1}^{n} t_i H_{\text{chp},i} \left(p_{\text{e}} \frac{\eta_{\text{e}}}{\eta_{\text{heat}}} - p_{\text{gas}} \left(\frac{1}{\eta_{\text{heat}}} - \frac{1}{\eta_{\text{ref}}}\right)\right) - kC + \text{constant} \\
&= b \sum_{i=1}^{n} t_i H_{\text{chp},i} - kC + \text{constant} \\
&= b \sum_{i=1}^{n} t_i \min(C, H_i) - kC + \text{constant}
\end{aligned} \tag{B.1}$$

Apart from the constant, this is essentially the same as our B in (2.3) above.

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