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A comparison of catching-up premium rate models

This paper discusses and compares two models for the catching-up premium rate, a partial adjustment (PA) model and a linear quadratic regulator (LQR) model. The models are different in that the PA model is a solution to a static optimisation problem, while the optimisation problem in the LQR model is dynamic. With respect to the economic principle of premium smoothing, it turns out that the LQR model is the preferable model. In addition, the simulation outcomes of this model are more consistent with the institutions of the Dutch pension system.

Contents

1 Introduction

The total pension premium rate consists of two components, the contribution rate and the catching-up premium rate. The contribution rate finances the accrual of pension rights while the catching-up premium finances (possible) wealth deficits of a pension fund. The contribution rate and the catching-up premium rate have a different effect on the economy. From an economic point of view, the contribution rate can be interpreted as delayed income needed to finance the period after retirement. In that sense, pension accrual payments are savings which do not have negative effects on the economy, as they do not distort individual labour supply decisions.

Catching-up payments however, can be considered as distorting taxes that affect labour supply. These payments do not yield direct benefits to an employer and employee. Consequently, an employer could decide to demand less labour while an employee may reduce its amount of labour supply, ultimately resulting in lower employment and production. Similar to the economic principles regarding normal taxation, the distorting effect of catching-up payments can be minimized by smoothing these payments over time.

In this paper the contribution rate is assumed to be exogenous and the main focus is on the catching-up premium rate. We will compare two models for the catching-up premium rate, a partial adjustment model (PA) and a linear quadratic regulator (LQR) model. Special attention will be given to the ability of the models to smooth premium payments over time. In both models the premium rule is an optimal solution to a minimisation problem. The models are different in that the PA model faces a static optimisation problem while in the LQR model this minimisation problem is dynamic. That means, the pension fund takes the future situation into account when it determines the present premium rate.

The models will be compared in two ways. First, we investigate the short-term effects of a shock to the pension system using a simple partial derivative analysis. Throughout this paper we are concerned with two sort of shocks: shocks that affect the effective rate of return of a pension fund (like a stock market crash) and shocks that affect the growth rate of pension rights (like a change in survival probabilities). Second, we will compare the transition effects of a shock in both models using a graphical impulse response analysis.

We find that the forward looking LQR model is preferable over the static PA model. Due to its forward looking character this model better smoothes catching-up payments over time. In the PA model, a shock to the pension system is largely absorbed by short-lived premium rate changes. In addition, it turns out the outcomes of the LQR model are more consistent with the institutions of the Dutch pension system.

The rest of the paper is organized as follows. Section 2 presents the PA model and the LQR model. Section 3 describes the partial derivative analysis and section 4 presents the impulse response analysis. Finally, section 5 concludes.

2 Modelling the catching-up premium

In both the PA model and the LQR model the catching-up premium rate is an optimal solution to a loss function, although the specification of this function differs between the models. Consistent with the instructions of the Dutch pension supervisor, the Pensioen- en Verzekeringskamer (PVK), the models impose that the funding ratio converges to a target value. The PVK prescribes that a pension fund whose funding ratio is lower than the required level, must reduce its deficit within a period of at most fifteen years.[1] Before we discuss the models in detail, we first introduce some definitions that are frequently used in this paper.

2.1 Definitions

We assume that the funding ratio (D) and the asset holdings of the pension fund (W) are measured at the end of the period, i.e., when all premium payments have been collected. So, we define,

$$D_t = \frac{W_t}{R_t} \tag{2.1}$$

$$W_t = (1+r_t)W_{t-1} + \varsigma_t G_t + \tau_t G_t - B_t \tag{2.2}$$

where R are pension liabilities, G is the premium base, B pension benefits, r the effective rate of return to pension wealth, ς the contribution rate and, finally, τ the catching-up premium rate. We assume that the pension rights grow according to,

$$\frac{R_t}{R_{t-1}} = (1+g_t)(1+\psi_t)^{\sigma_t}, \quad \psi_t > 0, \;\; 0 < \sigma_t \leqq 1 \tag{2.3}$$

with g the growth rate of pension liabilities due to factors like, for example, population growth and ψ is the indexation factor with respect to wages and prices. We allow for indexation discounts since $0 < \sigma \leqq 1$. Combining equation (2.1), (2.2) and (2.3) we rewrite the law of motion of the funding ratio as follows,

$$D_t = (1+k_t)D_{t-1} + P_t + A_t \tag{2.4}$$

[1] For a detailed overview of the most recent solvency instructions of the Dutch pension supervisor, see PVK (2004).

where k is the effective rate of return in terms of the funding ratio, P the catching-up premium receipts (in terms of the premium base) and A the autonomous part of the funding ratio. Hence,

$$\begin{aligned}
k_t &= \frac{(1+r_t)}{(1+g_t)(1+\psi_t)^{\sigma_t}} - 1 \\
P_t &= \tau_t \frac{G_t}{R_t} \\
A_t &= \varsigma_t \frac{G_t}{R_t} - \frac{B_t}{R_t}
\end{aligned} \tag{2.5}$$

2.2 Partial adjustment (PA) model

The general objective of a catching-up premium model is to smoothly close the gap between the current value of the funding ratio and its required level. The PA model considered here is an optimal solution to a loss function that imposes costs to fluctuations in both the funding ratio and the premium rate. Let us suppose that this loss function is quadratic and has the following form

$$L(t) = (D_t(\tau_t) - D^*)^2 + \lambda (D_t(\tau_t) - D_{t-1})^2 + \theta (P_t(\tau_t) - P_{t-1})^2, \quad \lambda > 0, \ \ \theta > 0 \tag{2.6}$$

where D^* is the required level of the funding ratio and assumed to be exogenous in this paper. To ensure that the funding ratio converges to this target value, equation (2.6) penalizes deviations from this target level. The second right-hand side term of (2.6) is the cost term associated with funding ratio fluctuations and reflects the institutional requirements imposed upon a pension fund. The weight that the fund (or implicitely, the pension authority) assigns to this term is constant and equal to λ. The third term of (2.6) represents the costs the pension fund assigns to year-to-year premium rate fluctuations. We thus assume that a pension fund realises that large premium rate fluctuations are welfare reducing for its participants. The weight that is assigned to this component is θ and is also time-invariant. Note that (2.6) is a static minimisation problem so that the fund does not take the future economic situation into account. From the first-order condition with respect to the catching-up premium we derive,

$$D_t - D^* + \lambda (D_t - D_{t-1}) + \theta (P_t - P_{t-1}) = 0 \tag{2.7}$$

Substituting the law of motion of the funding ratio (2.4) in (2.7), we obtain the following premium rate,

$$\tau_t = \frac{R_t}{G_t(1+\lambda+\theta)} \left(D^* - (1+k_t+\lambda k_t) D_{t-1} - (1+\lambda) A_t + \theta \frac{G_{t-1}}{R_{t-1}} \tau_{t-1} \right) \tag{2.8}$$

Equation (2.8) is the policy rule of the pension fund. This function is a weighted average of the target value of the funding ratio, the previous year's funding ratio, the autonomous part of the funding ratio and the lagged catching-up premium rate. As we may expect, the premium rate is increasing in the target funding ratio and decreasing in the autonomous development of the funding ratio. The intention of the fund to smooth the catching-up payments over time, is reflected by the lagged premium rate in the last term of (2.8).

The loss function is minimised if the funding ratio is constant and equal to its target value ($D_t = D_{t-1} = D^*$) and the premium receipts do not change anymore ($P_t = P_{t-1}$). Then the long-run premium rate is,

$$\tau_t = -\frac{R_t}{G_t}\left(k_t D^* + A_t\right) \tag{2.9}$$

So, in the long run the catching-up premium only depends on the wealth return ($k_t D^*$) and the autonomous development of the funding rate. In the hypothetical case in which the contribution rate is cost-effective and the effective rate of return is equal to the discount factor of pension rights, we have $A_t = -k_t D^*$ and consequently, the catching-up premium will be equal to zero.

The premium rate of equation (2.8) is a second-order difference equation. This can be shown by eliminating D from equation (2.8) ultimately leading to,

$$P_t = \frac{1}{1+\lambda+\theta}\left[\left(2\theta + k_t\theta + \lambda\right)P_{t-1} - \theta\left(1+k_t\right)P_{t-2} - \lambda\left(A_t - A_{t-1}\right) - k_t D^*\right] \tag{2.10}$$

In order to ensure that the funding ratio attains its required level, equation (2.10) requires that the premium rate must be stable in the long run. Stability implies that the long-run behaviour of the premium rate does not depend on initial conditions regarding financial wealth. Stability puts the following condition on θ,[2]

$$\theta < \frac{1+\lambda}{k_t} \tag{2.11}$$

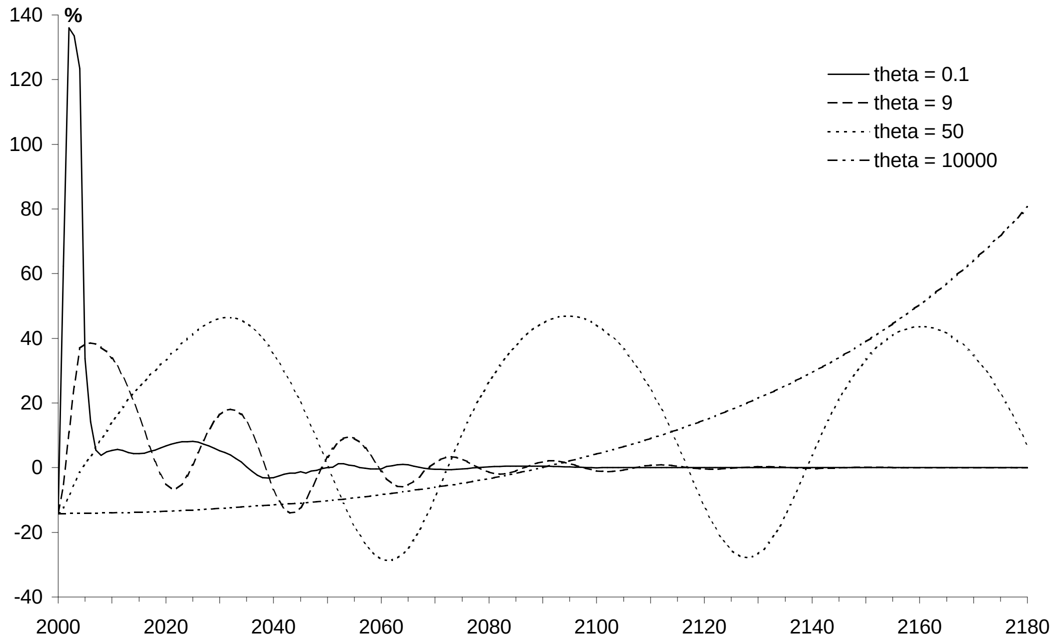
Besides the requirement of stability, there is more to care about. In general, a second-order difference equation can display oscillations, depending on whether the characteristic equation has complex roots or not. However, an oscillating premium rate is not compatible with the objective of the pension fund considered in this model. The pension fund tends to smooth catching-up payments, because it is generally assumed that this minimises the distorting effect of these payments on labour supply decision of its participants. To prevent that the characteristic equation of (2.10) has complex roots, a second condition is imposed upon λ and θ,

$$\left(2\theta + k_t\theta + \lambda\right)^2 > 4\theta\left(1+k_t\right)\left(1+\lambda+\theta\right) \tag{2.12}$$

Note that equation (2.12) is always satisfied if no cost is imposed upon premium rate fluctuations ($\theta = 0$).

[2] A solution to a second-order difference equation is stable if and only if the modulus of each root of the characteristic equation is less than one. In terms of the coefficients in the equation $x_t + \varphi_1 x_{t-1} + \varphi_2 x_{t-2} = c_t$, stability implies that $\varphi_2 < 1$ and $|\varphi_1| < 1 + \varphi_2$. In our case we can omit the second condition because it can be shown that this condition implies $\theta\left(1+k_t\right) - 1 < \theta\left(1+k_t\right)$, which is always satisfied because it is assumed that $\theta > 0$ and $1 + k_t > 0$.

Figure 2.1 Sensitivity of the premium rate to changes in θ



To illustrate the oscillating behaviour of equation (2.10), figure 2.1 shows the premium rate for different values of θ. For simplicity we only impose costs to premium rate fluctuations, that is, we set $\lambda = 0$ in equation (2.6). This restriction implies for the stability condition (2.11) and real root condition (2.12) respectively,

$$\theta < \frac{1}{k_t} \tag{2.13a}$$

$$\theta > \frac{4\left(\frac{1}{k_t}+1\right)}{k_t} \tag{2.13b}$$

Note that the right-hand side of inequality (2.13b) is strictly larger than the right-hand side of (2.13a). Therefore, the premium rate converges if and only if it contains oscillations. This is exactly what we observe in figure 2.1. When θ is low (high) the rate of convergence is high (low) and the oscillations are less (more) pronounced. Because we assume that $k = 0.019$, the premium rate does not converge anymore if θ exceeds the value of 50.[3] In addition, if θ is larger than about 10^4, the premium rate is free from oscillations and becomes an exponential function.

[3] In section 4 we will discuss the calibration of the parameters.

2.3 Linear quadratic regulator (LQR) model

In the PA model the determination of the catching-up premium rate is a static decision. In the LQR model we relax this assumption and assume that the pension fund takes account of the future economic situation at the time the current premium rate is determined.

The general idea of the LQR model is to a large extent identical to the PA model. That means, the LQR model minimises a quadratic loss function subject to a linear transition function. Solving the Bellman equation of the loss function gives an optimal policy rule for the control variable as a linear function of the state variable. In the loss function, there is one state variable, the funding ratio, and one control variable, the catching-up premium.[4] We will not use levels of the funding ratio and catching-up premium rate in the loss function, though, but deviations of these variables from their long-term values.

In the long run the funding ratio has to be equal to its target value ($D_t = D^*$) and for the long-term catching-up premium we have,

$$\overline{P} = -\overline{A} - \overline{k}D^* \tag{2.14}$$

where a bar above a variable denotes the equilibrium value which is assumed to be constant in the long run. The state variable (X) and control variable (U) can be written as,

$$X_t \equiv D_t - D^* = (1 + k_t)X_{t-1} + U_t + Z_t \tag{2.15}$$

$$U_t \equiv P_t - \overline{P} = P_t + A_t + \overline{k}D^* \tag{2.16}$$

with $Z_t \equiv D^* \left(k_t - \overline{k}\right)$. Since equation (2.16) is defined in deviation from the long-run premium rate, shocks to k do not directly enter the control variable. Suppose, the representative pension fund has the following loss function,

$$L_0(X_0) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left(\lambda X_t^2 + \theta U_{t+1}^2\right), \quad \rho > 0,\ \lambda > 0,\ \theta > 0 \tag{2.17}$$

where ρ is the pure rate of time preference and $\theta = \pi/(1+\rho)$ is the discounted version of the cost parameter π. The pension fund assigns costs to deviations of the funding ratio and premium rate from their long-run values with, respectively, weight λ and θ. For convenience, we use the same symbols for the cost parameters as in the PA model although their interpretation is slightly different.

From (2.17) we observe that the pension fund is not only concerned with the present situation, but also discounts future deviations of the state and control variable. In addition, the

[4] For a description of the general deterministic and stochastic LQR model in matrix notation, we refer to, respectively, chapter 4 of Ljungqvist and Sargent (2004) and chapter 2 of Heer and Maussner (2005).

observed loss in period t depends on the *current* state and the (expectation of the) *one period ahead* control instead of the current control.[5] The reason for this is that the value of the current control is already captured by the current state, see equation (2.16).

Now the idea of the LQR model is the following. Given the law of motion of equation (2.15), the pension fund chooses the sequence $\{U_t\}_{t=0}^{\infty}$ in such a way that the loss function becomes minimal. In appendix A it is derived that the optimal premium rate satisfies,

$$\tau_t = \frac{R_t}{G_t}\left[-\frac{1}{(\theta+\theta\rho+\alpha_t)}\left(\alpha_t(1+k_t)D_{t-1}-\alpha_t\left(1+\bar{k}\right)D^*+\frac{1}{2}\beta_t\right)-\bar{A}-\bar{k}D^*\right] \tag{2.18}$$

Equation (2.18) has a lot of similarities with the PA premium rule. The catching-up premium rate negatively depends on the previous year's funding ratio and the autonomous accumulation of financial wealth, and positively on the target value of the funding ratio. Note that the lagged premium rate does not enter equation (2.18). Consequently, the LQR premium rate is not subject to oscillations.

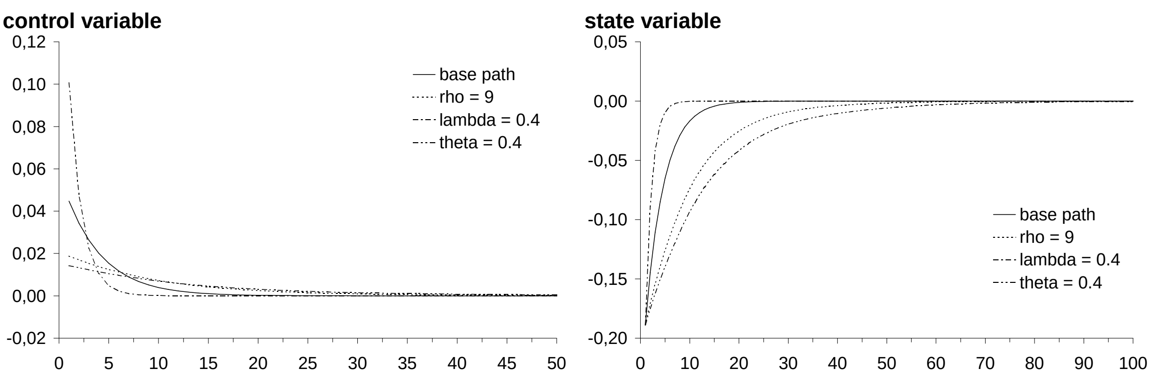
The forward looking character of the LQR premium rate is captured by the time dependent coefficients α and β. As derived in appendix A, these coefficients are functions of the one period ahead effective rate of return. That is,

$$\alpha_t = \lambda + \frac{\alpha_{t+1}}{1+\rho}(1+k_{t+1})\left(1-\frac{\alpha_{t+1}(1+k_{t+1})}{(1+\rho)(\theta+\theta\rho+\alpha_{t+1})}\right) \tag{2.19a}$$

$$\beta_t = \frac{1}{1+\rho}(1+k_{t+1})(2\alpha_{t+1}Z_{t+1}+\beta_{t+1})\frac{\theta(1+\rho)}{(\theta+\theta\rho+\alpha_{t+1})} \tag{2.19b}$$

In appendix A we also show under which condition β approaches zero in the long run. If this condition is met, the funding ratio will reach its target value and equation (2.18) will reduce to its equilibrium value given by equation (2.14).

Figure 2.2 Sensitivity of the state and control variable



[5] We assume that agents have perfect foresight and therefore exactly know future values of the premium rate. This assumption is consistent with the GAMMA model.

The time preference rate (ρ) and the cost parameters (λ and θ) together determine the rate of convergence of the state variable (X) and the control variable (U). Figure 2.2 shows the sensitivity of the control variable (left panel) and state variable (right panel) to changes in one of these parameters. In the base situation λ and θ are set to 0.1 and ρ to 3.

Consider first an increase of ρ to 9. From the left panel of figure 2.2 we see that the control variable initially drops down relative to the base situation and then more gradually declines towards zero. Consequently, the convergence rate of the state variable also declines and it takes more time until the funding ratio has converged to its equilibrium level (see the right panel of figure 2.2. The interpretation is as follows. When ρ increases, the (short-sighted) pension fund shifts the premium burden needed to reduce the wealth deficit to the future.

Increasing the parameter λ leads to the opposite effect. Recall that a higher value of λ places more costs to deviations of the funding ratio from its equilibrium value. Therefore, when λ increases, the rate of convergence of the funding ratio increases, but this does not occur costless. Relative to the base situation, the control variable increases at impact and then quickly declines towards zero.

The effect of an increase in θ actually leads to the same conclusions as a rise in ρ. If θ is large, the pension fund imposes much costs to deviations of the premium rate from its equilibrium value. As a consequence, the level of the control variable u_t initially drops down as the convergence rate of the funding ratio does.

3 Comparing the models

In this section the PA model and the LQR model are compared. We will investigate how different shocks affect the funding ratio and premium rate in the shock period. In the next section we will discuss the transition effects. We focus on two types of shocks, a shock to the asset holdings of a pension fund and a shock to its liabilities. We will also consider the role of cutting down indexation promises. It is assumed that the shocks occur before the catching-up premium rate has been fixed.

3.1 Shock to asset holdings

A shock to the asset holdings of a pension fund can take different forms, like for example an asset market crash or an employer's contribution. In this paper asset shocks have in common that they affect the effective rate of return (r). To analyse the effect of such a shock on the funding ratio and premium rate, we simply compute the corresponding partial derivatives with respect to

r. Using equation (2.8) the partial derivative of τ_t with respect to r_t for the PA model is,

$$\text{PA} \quad \frac{\partial \tau_t}{\partial r_t} = -\frac{R_{t-1}}{G_t}\left(\frac{1+\lambda}{1+\lambda+\theta}\right)D_{t-1} < 0 \tag{3.20}$$

which is strictly negative as intuitively expected. If the portfolio return declines (increases), the pension fund has to increase (decrease) the premium rate in order to bring the funding ratio in the direction of its required level. Equation (3.20) is increasing in θ. Recall that θ determines the costs that are imposed upon premium rate fluctuations. When this cost is high, the increase (decline) of the premium rate in reaction to a negative (positive) asset shock is low.

It is important to know how the impulse of the rate of return influences the funding ratio. Substituting equation (2.8) in the law of motion of the funding ratio (2.4) and taking the partial derivative of d_t with respect to r_t, we get,

$$\text{PA} \quad \frac{\partial D_t}{\partial r_t} = \frac{\theta}{(1+g_t)(1+\psi_t)^{\sigma_t}(1+\lambda+\theta)}D_{t-1} \geqq 0 \tag{3.21}$$

Let us first analyse the two most extreme situations, i.e., in which no cost is placed upon premium rate fluctuations ($\theta = 0$) or to funding ratio fluctuations ($\lambda = 0$). If there are no costs associated with premium rate movements, this rate will be adjusted immediately after a shock and there will be no effect on the funding ratio. Of course, this result is not incompatible with the instructions of the PVK, but is is far from realistic. In many practical situations a shock will affect the funding ratio like, for example, an employer's contribution with the intention to improve the funding ratio. Imposing no costs upon funding ratio fluctuations is also not very realistic. From equation (3.21) we observe that for $\lambda = 0$ the change of the funding ratio after a shock is maximal. However, in reality it is not reasonable that the funding ratio behaves as a jump variable. We therefore only consider the situation in which both $\lambda > 0$ and $\theta > 0$.

Penalizing both premium rate and funding ratio fluctuations brings the PA model closer to reality. If $\lambda > 0$ and $\theta > 0$, equation (3.20) is strictly negative and equation (3.21) strictly positive. This implies that a positive (negative) shock to asset holdings induces both the premium rate to decrease (increase) and the funding ratio to increase (decrease).

For the LQR model, the partial derivatives of the premium rate and funding ratio with respect to r are respectively,

$$\text{LQR} \quad \frac{\partial \tau_t}{\partial r_t} = -\frac{R_{t-1}}{G_t}\left(\frac{\alpha_t}{\theta+\theta\rho+\alpha_t}\right)D_{t-1} < 0 \tag{3.22}$$

$$\text{LQR} \quad \frac{\partial D_t}{\partial r_t} = \frac{\theta(1+\rho)}{(\theta+\theta\rho+\alpha_t)(1+g_t)(1+\psi_t)^{\sigma_t}}D_{t-1} > 0 \tag{3.23}$$

The behaviour of the LQR model is quite comparable with the PA model.[6] A positive (negative) asset shock leads to a decline (increase) in the premium rate and to an improvement

[6] In equation (3.22) and (3.23), but also in equation (3.25) and (3.27), it is assumed that $\alpha_t > 0$ for all t. In appendix A it is shown under which condition this assumption holds.

(deterioration) of the funding ratio. Equation (3.22) and (3.23) are both increasing in θ. Therefore, imposing a high (low) cost to deviations of the catching-up premium rate from its equilibrium level leads to a small (large) response of the premium rate and the impact on the funding ratio is high (low).[7]

Although the cost parameter λ does not appear in (3.22) and (3.23), from equation (2.19a) we observe that there is direct positive link between α and λ. Equations (3.22) and (3.23) are both decreasing in α. Thus, if more costs are assigned to funding ratio deviations (λ increases), the burden of the shock will shift from the funding ratio to the premium rate.

3.2 Shock to liabilities

Shocks to the liability side of a pension fund also occur in reality. For instance, it is possible that the full retirement age is increased or that the interest rate by which the liabilities are discounted changes. At impact, all these shocks affect the growth rate of the liabilities (g). It may be expected that an increase (decrease) in g *ceteris paribus* deteriorates (improves) the funding ratio. To analyse what actually happens in the PA and LQR model, we compute the partial derivative of d_t with respect to g_t,

$$\text{PA} \quad \frac{\partial D_t}{\partial g_t} = -\frac{\theta}{(1+\lambda+\theta)(1+g_t)^2(1+\psi_t)^{\sigma_t}}\left[(1+r_t)D_{t-1} + \frac{M_t}{R_{t-1}}\right] \lesseqqgtr 0 \qquad (3.24)$$

$$\text{LQR} \quad \frac{\partial D_t}{\partial g_t} = -\frac{\theta(1+\rho)(1+r_t)}{(\theta+\theta\rho+\alpha_t)(1+g_t)^2(1+\psi_t)^{\sigma_t}} D_{t-1} < 0 \qquad (3.25)$$

where $M_t \equiv \zeta_t G_t - U_t$ in (3.24). As intuitively clear, a positive shock to the growth rate of the liabilities has the same effect as a negative asset shock. That means, if this growth rate increases, the funding ratio will be negatively affected in both models. In the PA model, we observe that the burden of the shock shifts to the funding ratio (premium rate) if more weight is given to θ (λ). The same holds for the LQR model. That is, the response of the funding ratio to a shock is increasing (decreasing) in θ (λ).

3.3 Effectiveness of indexation discounts

It is important to investigate the effectiveness of cutting down indexation promises in the PA and LQR model. Usually the pension rights of the participants of a pension fund will be indexed by the inflation and productivity rate. However, a pension fund which is in state of under-capitalization can use indexation discounts as additional instrument to improve its funding ratio. Therefore, it is necessarily that cutting down indexation improves the funding ratio in the

[7] Note from equation (2.19a) that θ also influences α, but this effect is rather small.

models. The discount rate is given by $1-\sigma$. In order to get a positive effect of indexation discounts on the funding ratio, the partial derivative of D_t with respect to σ_t must be negative. Let us check this for the PA and LQR model,

$$\text{PA} \quad \frac{\partial D_t}{\partial \sigma_t} = -\frac{\theta \ln(1+\sigma_t)}{(1+\lambda+\theta)(1+g_t)(1+\psi_t)^{\sigma_t}}\left[(1+r_t)D_{t-1}+\frac{M_t}{R_{t-1}}\right] \tag{3.26}$$

$$\text{LQR} \quad \frac{\partial D_t}{\partial \sigma_t} = -\frac{\theta(1+\rho)(1+r_t)\ln(1+\sigma_t)}{(\theta+\theta\rho+\alpha_t)(1+g_t)(1+\psi_t)^{\sigma_t}}D_{t-1} < 0 \tag{3.27}$$

Assuming that the term in square brackets is positive, equation (3.26) is indeed negative. From equation (3.27) we notice that the same holds for the LQR model. Thus in both models indexation discounts are effective.

Summarizing, the initial effect of an asset shock and liability shock on the funding ratio and premium rate are qualitatively quite comparable in both models. That means, a positive asset shock, or equivalently, a negative liability shock, improves the funding ratio but makes it also possible to lower the premium rate. In addition, cutting down indexation is an effective instrument for a pension fund to improve its financial situation.

4 Simulations

We emphasize that the analysis above only investigates the initial impact of a shock to the pension system. In this section we will analyse the transition effects. More specifically, we will graphically analyse the effect of an employer's contribution and a stock market crash. We will first discuss the calibration of the model parameters and show the baseline path of the catching-up premium rate and funding ratio in both models.

4.1 Data and calibration

For simulation purposes, we need data about the contribution rate (τ), the premium base (G), pension liabilities (R), pension benefits (B), the effective rate of return (k) and an initial level of asset holdings (W). We will use artificial data generated by the dynamic general equilibrium model GAMMA.[8] From these data we deduct the long-term growth rates of the premium base ($\overline{h}$) and of the pension rights ($\overline{g}$) and the equilibrium value of the effective rate of return ($\overline{k}$). The full set of calibrated model parameters is presented in table 4.1.

The calibration of the parameters λ, θ and ρ needs some explanation because these parameters determine the adjustment process of the funding ratio. In the PA model, a high (low)

[8] For a description of the GAMMA model, see Draper et al. (2005).

Table 4.1 Calibration

model dependent parameters	PA	LQR	model independent parameters	
λ	12	0.3	$\bar{k}$	0.019
θ	8	0.1	$\bar{h}$	0.035
ω	0.4	-	$\bar{g}$	0.035
ρ	-	4	$\bar{r}$	0.055
η	10	-	D^*	1.5

λ imposes high (low) costs to year-to-year funding ratio fluctuations so that this ratio converges slowly (quickly). A low (high) value for θ, the cost parameter associated with premium rate fluctuations, leads to the same effect. In the LQR model, the adjustment process of the funding ratio is not only determined by the penalty coefficients λ and θ, but also by the rate of time preference ρ. A high (low) ρ decreases (increases) the convergence rate of the funding ratio in the LQR model.

The PVK prescribes that a pension funds with unfunded liabilities must reduce its deficit within a period of at most fifteen years. For a pension fund that only guarantees a nominal pension claim, the PVK employs a target funding ratio of 130 percent.[9] For the PA model this requirement is met if $\lambda = 12$ and for the LQR model this is satisfied for $\lambda = 0.3$, $\theta = 0.1$ and $\rho = 4$.

The calibration of θ in the PA model is determined by the stability condition (equation (2.11)) and the real root condition (equation (2.12)). We set $\theta = 8$. With this value the real root and stability condition is satisfied and the adjustment process of the funding rate is consistent with the instruction of the PVK.

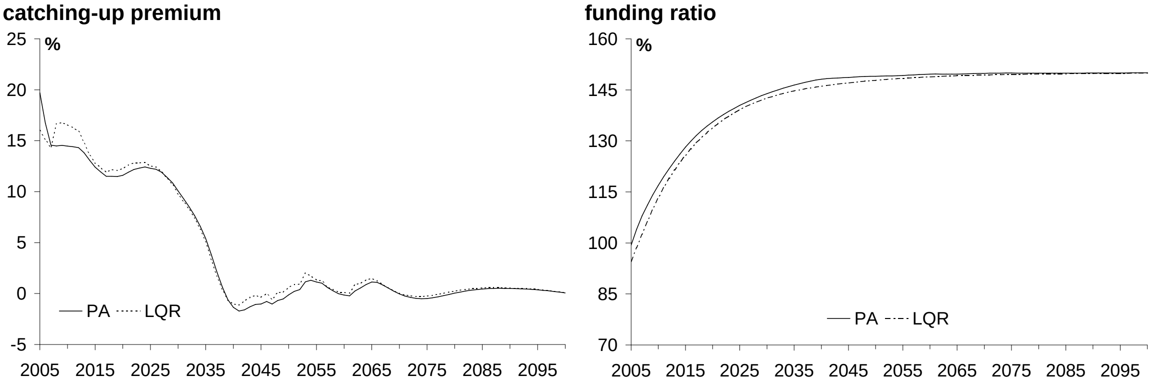
To prevent that the catching-up premium of the PA model becomes unrealistically high, equation (2.8) is truncated by the following function:

$$\widetilde{\tau}(t) = \begin{cases} \left[(\omega - \varsigma_t)^{-\eta} + \tau_t^{-\eta}\right]^{-1/\eta}, \text{ if } \tau_t > 0 \\ \\ \tau_t, \text{ if } \tau_t \leqq 0. \end{cases} \tag{4.1}$$

where ω is the upper-bound of the total premium rate rate (contribution rate plus catching-up premium rate) and assumed to be 40 percent. In addition, η is a smoothing parameter which has a value of 10.

[9] In accordance with GAMMA, we set $D^* = 1.5$. With this choice we implicitely assume that the pension fund guarantees a wage indexed pension claim.

Figure 4.1 Baseline path



4.2 Baseline path

The baseline path of the catching-up premium rate and funding ratio is represented in figure 4.1. In the starting position the representative pension fund is in state of considerable under-capitalization, that is $D_t < D^*$. We observe that in the first forty years the premium rate is above average, until the funding ratio reaches its target value. In the long run the premium rate converges to zero.

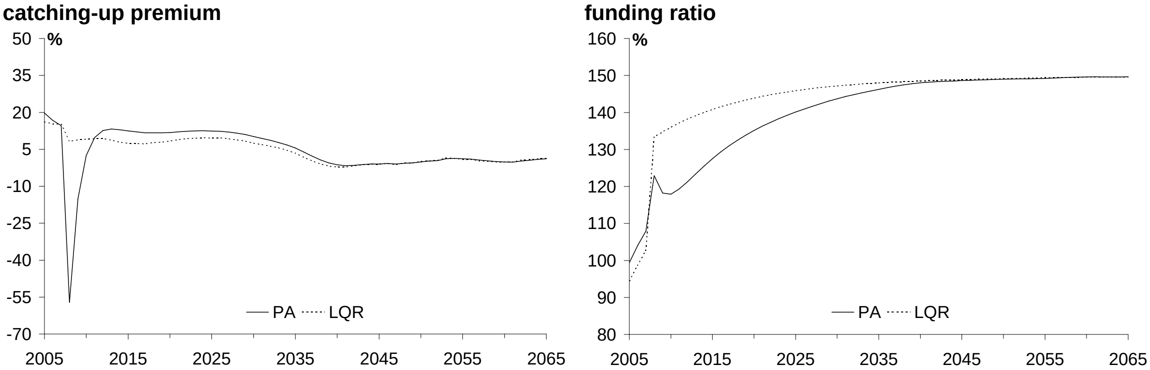
In the absence of shocks, the similarity of the baseline paths of the PA model and LQR model is extremely large. Of course, to a large extent this is due to the way the model parameters are calibrated. In both models the funding ratio equals 130 percent after fifteen years, which is consistent with the PVK instruction.

4.3 Employer's contribution

Now we will consider the impact of an employer's contribution to a pension fund which is in state of under-capitalization. The employer's contribution takes place in 2008 and is implemented by a one-time increase of the effective rate of return from 5.5 percent to 35 percent.

The impact of this shock on the contribution rate and funding ratio is graphically presented in figure 4.2. The most striking fact is the enormous decline of the premium rate in the PA model. A large part of the employer's contribution is substituted back to the fund participants through negative premium rates. As a consequence, the funding ratio improves just moderately. The funding rate increases in 2008 by 14 percent which is 11 percent-points higher than if there was no shock. After its initial increase in 2008, the funding ratio drops down in 2009 to a level of 118 percent. The reason of this implausible result is connected with the second-order character of the PA policy rule.

Figure 4.2 Employer's contribution



The improvement of the funding ratio could be increased by declining the cost imposed upon funding ratio changes (low λ) and/or increasing the penalty imposed upon premium rate fluctuations (high θ). However, this will increase the possibility of unacceptable oscillations.

In the LQR model the control variable does not directly depend on an employer's contribution. Therefore the premium rate just slightly declines and the funding ratio increases significantly. The funding ratio increases with 29 percent in 2008 to a level of 133 percent, which is 25 percent-points higher than in the absence of an employer's contribution. After this initial jump the capitalisation rate gradually converges to its required level of 150 percent.

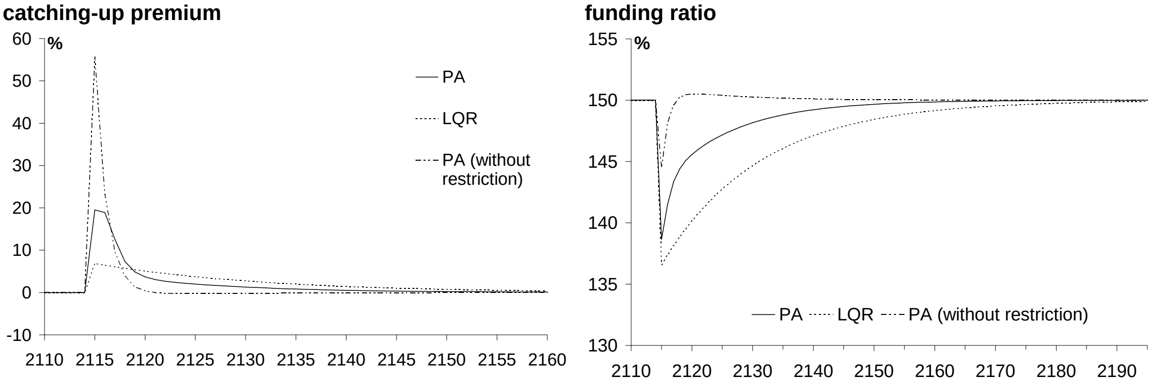
4.4 Stock market crash

We now consider the effect of a stock market crash. Contrary to the employer's contribution, we assume that the pension fund is in equilibrium, that is $D_t = D^*$. The crash takes place in the year 2115 and reduces the effective rate of return with one percent-point (see figure 4.3).

At impact the funding ratio decreases in both models. However, the magnitude of the premium rate changes differs sharply. In the PA model the premium rate rises to about 20 percent while in the LQR model this level is 7 percent. From then on, the PA premium rate quickly declines and after four years this rate is lower than the gradually declining LQR premium rate. From figure 4.3 (left panel) we may conclude that the LQR premium rate does a better job to smooth distorting catching-up payments. As a consequence, the funding ratio quicklier attains its target value in the PA model relative to the LQR model.

To illustrate the importance of the maximisation restriction in the PA model, figure 4.3 also contains the funding ratio and premium rate paths if equation (4.1) is not imposed. In this situation the premium rate becomes unrealistically high and the effect of the shock on the funding ratio is rather limited.

Figure 4.3 Stock market crash



5 Conclusion

In this paper we have discussed two types of catching-up premium models, a static partial adjustment (PA) model and a dynamic linear quadratic regulator (LQR) model. In both models the catching-up premium is an optimal solution to a quadratic loss function that imposes costs upon premium and funding ratio fluctuations. The definition of fluctuations differs between the models due to the fact that the optimisation problems are not equal. The PA model minimises year-to-year fluctuations while the LQR model minimises deviations of the control and state variable from their equilibrium levels.

The models are formally compared using a partial derivative analysis and graphically using impulse response figures. For a number of reasons we conclude that the LQR model is preferable over the PA model. First, due to its forward looking character, the LQR model is a better tool to smooth catching-up payments over time. Second, the simulation outcomes of the LQR model are more realistic. For example, in the PA model a stock market crash causes a large increase in the premium rate. Consequently, the decrease of the funding ratio is too small. Although this result is not incompatible with the PVK rules, in practice a pension fund will smooth the effects of a shock over a longer horizon. Also, an employer’s contribution does not significantly improve the funding ratio in the PA model, in contrary to the LQR model. Third, the PA premium rule is a second-order difference equation which is sensitive to implausible oscillations. We observe that this sensitivity increases with the penalty imposed upon premium rate fluctuations. The LQR rule is not susceptible to this problem.

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Appendix A Derivation of the LQR model

The LQR minimisation problem (2.17) can be written as a Bellman equation in the following way,

$$L_{t-1}\left(X_{t-1}\right)=\min_{U_t}\left\{\lambda X_{t-1}^2+\theta U_t^2+\frac{1}{1+\rho}L_t\left(X_t\right)\right\} \tag{A.1}$$

We guess that the solution to the infinite horizon problem is of the form $L_t(X_t)=\alpha_t X_t^2+\beta_t X_t+\gamma_t$, where α, β and γ are time dependent coefficients. Substituting this guess in (A.1) and using the transition law of the funding ratio to eliminate next period's state, the Bellman equation becomes,

$$L_{t-1}(X_{t-1})=\min_{U_t}\left\{\begin{array}{c}\left(\lambda X_{t-1}^2+\theta U_t^2\right)+\frac{\alpha_t}{1+\rho}\left[(1+k_t)X_{t-1}+U_t+Z_t\right]^2+\\ \frac{\beta_t}{1+\rho}\left[(1+k_t)X_{t-1}+U_t+Z_t\right]+\frac{\gamma_t}{1+\rho}\end{array}\right\} \tag{A.2}$$

Differentiating (A.2) with respect to the control U_t leads to,

$$\begin{aligned}\frac{\partial L_{t-1}(X_{t-1})}{\partial U_t}&=2\theta U_t+\frac{2\alpha_t}{1+\rho}\left[(1+k_t)X_{t-1}+U_t+Z_t\right]+\frac{\beta_t}{1+\rho}=0\\ U_t&=-\frac{1}{1+\rho}\left(\theta+\frac{\alpha_t}{1+\rho}\right)^{-1}\left(\alpha_t(1+k_t)X_{t-1}+\alpha_t Z_t+\frac{1}{2}\beta_t\right)\end{aligned} \tag{A.3}$$

so that the optimal policy rule for the catching-up premium is,

$$\tau_t=\frac{R_t}{G_t}\left[-\frac{1}{(\theta+\theta\rho+\alpha_t)}\left(\alpha_t(1+k_t)D_{t-1}-\alpha_t\left(1+\overline{k}\right)D^*+\frac{1}{2}\beta_t\right)-\overline{A}-\overline{k}D^*\right] \tag{A.4}$$

The coefficients α, β and γ can be derived by substituting (A.3) in (A.2) and comparing the quadratic, linear and constant forms on both sides. Using standard algebra this ultimately gives the following recursive expressions, known as the Riccati recursions,

$$\alpha_{t-1}=\lambda+\frac{\alpha_t}{1+\rho}\left(1+k_t\right)\left(1-\frac{\alpha_t\left(1+k_t\right)}{\left(1+\rho\right)\left(\theta+\theta\rho+\alpha_t\right)}\right) \tag{A.5a}$$

$$\beta_{t-1}=\frac{1}{1+\rho}\left(1+k_t\right)\left(2\alpha_t Z_t+\beta_t\right)\frac{\theta\left(1+\rho\right)}{\left(\theta+\theta\rho+\alpha_t\right)} \tag{A.5b}$$

$$\gamma_{t-1}=-\frac{1}{\left(1+\rho\right)\left(\theta+\theta\rho+\alpha_t\right)}\left(\alpha_t Z_t+\frac{1}{2}\beta_t\right)^2+\frac{1}{1+\rho}\left(\alpha_t Z_t^2+\beta_t Z_t+\gamma_t\right) \tag{A.5c}$$

Recall that the results of section 3 are conditioned on the assumption that $\alpha_t>0$ for all t. Since by definition $\rho>0$, $\lambda>0$ and $\theta>0$, from equation (A.5a) it follows that $\alpha_t>0$ for all t if,[10]

$$\left(1+\rho\right)\left(\theta+\theta\rho+\alpha_t\right)>\alpha_t\left(1+k_t\right) \tag{A.6}$$

[10] It is also assumed that $\alpha_N=0$, where N denotes the last observation of the sample.

In our data $\rho > k_t$ for all t (see table 1), so equation (A.6) is satisfied. In addition, in order to ensure that $U_t = P_t - \overline{P} = 0$, we must have $\beta = 0$ in the long run. Since equation (A.5b) is a first-order difference equation, this condition is met if

$$\begin{aligned} \frac{\theta\,(1+k_t)}{\theta\,(1+\rho)+\alpha_t} &< 1 \\ 1+k_t &< 1+\rho+\frac{\alpha_t}{\theta} \end{aligned} \tag{A.7}$$

Again, this equation is satisfied if $\rho > k_t$ for all t.

Appendix B Symbols

A autonomous part of funding ratio

B pension benefits

D funding ratio

D^* target funding ratio

G premium base

g growth rate of pension liabilities

h growth rate of premium base

k effective rate of return of funding ratio

M autonomous part of asset holdings pension fund

P catching-up premium receipts (% pension liabilities)

R liabilities pension fund

r effective rate of return of pension wealth

U control variable (LQR model)

W asset holdings pension fund

X state variable (LQR model)

α Riccati recursion of x^2 (LQR model)

β Riccati recursion of x (LQR model)

γ Riccati recursion of constant term (LQR model)

ζ contribution rate (% premium base)

η smoothing parameter (PA model)

θ (discounted) cost parameter of premium rate fluctuations

λ cost parameter of funding ratio fluctuations

π cost parameter of premium rate fluctuations (LQR model)

ρ rate of time preference (LQR model)

σ indexation discount parameter

τ catching-up premium rate (% premium base)

ψ indexation parameter

ω maximal premium rate (PA model)